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REGULARITY RESULTS AND SOLUTION SEMIGROUPS FOR RETARDED FUNCTIONAL DIFFERENTIAL EQUATIONS

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We show that the solutions of the retarded functional differential equations in a Banach space, whose existence and uniqueness are established in paper of A. Favini and H. Tanabe, have some further regularity properties if the initial data and the inhomogeneous term satisfy some smootheness assumptions. Some results on the solution semigroups analogous to the one of G. Di Blasio, K. Kunisch and E. Sinestrari and to the one of E. Sinestrari are also obtained.

Keywords: retarded functional differential equation; regularity of solutions; analytic semigroup; solution semigroup; C_0-semigroup; infinitesimal generator.

Introduction

We consider the following retarded functional differential equation in a complex Banach space X:

$$\begin{cases} \dfrac{d}{dt}u(t) = Au(t) + A_1u(t-r) + \displaystyle\int_{-r}^{0} a(s)A_2u(t+s)ds + f(t), \ \ 0 \le t \le T, \\ u(0) = \varphi_0, \quad u(s) = \varphi_1(s) \ \text{ a.e. } \ s \in (-r, 0). \end{cases} \tag{0.1}$$

We assume that A is a densely defined closed linear operator which generates an analytic semigroup $e^{tA}, t \ge 0$, in X. Suppose $0 \in \rho(A)$ for simplicity. A_1 and A_2 are closed linear operators in X such that $D(A_1) \supset D(A)$, $D(A_2) \supset D(A)$, and a is a complex valued function defined in the interval $[-r, 0]$ such that $a \in L^1(-r, 0; \mathbb{C})$.

The following theorems which are improvements of the results by G. Di Blasio and A. Lorenzi [1] are established in A. Favini and H. Tanabe [2]:

Theorem A *Suppose* $0 < \theta < 1/p$. *If the following assumption is satisfied*:

(I) $\varphi_0 \in (X, D(A))_{\theta+1-1/p,p}, \ \varphi_1 \in W^{\theta,p}(-r, 0; D(A)), \ f \in W^{\theta,p}(0, T; X),$

then, there exists a unique solution u *of* (0.1) *satisfying*

$$u \in W^{\theta,p}(0, T; D(A)) \cap C([0, T]; (X, D(A))_{\theta+1-1/p,p}), \tag{0.2}$$

$$du/dt \in W^{\theta,p}(0, T; X). \tag{0.3}$$

Theorem B *Suppose* $1/p < \theta < 1$. *If the following assumption is satisfied*:

$$\text{(II)} \quad \begin{cases} \varphi_0 \in D(A), \quad \varphi_1 \in W^{\theta,p}(-r, 0; D(A)), \quad \varphi_1(0) = \varphi_0, \\ f \in W^{\theta,p}(0, T; X) \cap C([0, T]; (X, D(A))_{\theta-1/p,p}), \\ A\varphi_0 + A_1\varphi_1(-r) + \displaystyle\int_{-r}^{0} a(s)A_2\varphi_1(s)ds \in (X, D(A))_{\theta-1/p,p}, \end{cases}$$

then, problem (1.1) *admits a unique solution* u *such that*

$$u \in W^{\theta,p}(0,T;D(A)), \tag{0.4}$$

$$du/dt \in W^{\theta,p}(0,T;X) \cap C([0,T];(X,D(A))_{\theta-1/p}). \tag{0.5}$$

Note that since $A_2\varphi_1 \in C([-r,0];X)$, the integral $\int_{-r}^0 a(s)A_2\varphi_1(s)ds$ is well defined.

In this paper we prove further regularity of solutions u when φ_1 and f satisfy more regularity assumptions. Using this result we prove some results on solution semigroups analogous to the one of G. Di Blasio, K. Kunisch and E. Sinestrari [3] when $\theta < 1/p$, and to the one of E. Sinestrari [4] when $\theta > 1/p$. In case $\theta < 1/p$ it is shown that the map $\begin{pmatrix} \varphi_0 \\ \varphi_1 \end{pmatrix} \mapsto \begin{pmatrix} u(t) \\ u_t \end{pmatrix}$, where u is the solution of (0.1) with $f(t) \equiv 0$ and $u_t(s) = u(t+s), -r \le s \le 0$, is a C_0-semigroup in $(X,D(A))_{\theta+1-1/p,p} \times W^{\theta,p}(-r,0;D(A))$, and the characterization of its infinitesimal generater is given. This is nothing but a simple extention of a result of [3] in case where X is a Hilbert space and $\theta = 1$, $p = 2$. However, in case $\theta > 1/p$ the situation is a little more complicated. This is caused by the following fact. In this case there appears the space $(X,D(A))_{\theta+1-1/p,p}$ which is a subset of $D(A)$. If u belongs to this space, A_iu, $i = 1,2$, is defined, but may not belong to $(X,D(A))_{\theta-1/p,p}$. Therefore we assume the additional condition $A_iA^{-1} \in \mathcal{L}((X,D(A))_{\theta-1/p,p})$, $i = 1,2$. A comment on this assumption will be given in section 6. Under these hypotheses it will be shown that the map $\varphi_1 \mapsto u_t$ is a C_0-semigroup in $W^{\theta,p}(-r,0;D(A)) \cap C([-r,0];(X,D(A))_{\theta+1-1/p,p})$ with the characterization of its infinitesimal generater, where again u is the solution of (0.1) with $f(t) \equiv 0$ and $u_t(s) = u(t+s), -r \le s \le 0$.

For a Banach space Y we use the following norm of $W^{\theta,p}(0,T;Y)$:

$$\begin{aligned} N_{\theta,p,Y,(0,T)}(u) &= \left(\int_0^T \int_0^t \|u(t)-u(s)\|_Y^p (t-s)^{-1-\theta p} ds dt \right)^{1/p}, \\ \|u\|_{W^{\theta,p}(0,T;Y)} &= N_{\theta,p,Y,(0,T)}(u) + T^{-\theta}\|u\|_{L^p(0,T;Y)}. \end{aligned} \tag{0.6}$$

1. Regularity of Solutions: Case $\theta < 1/p$

First consider the case $0 < \theta < 1/p$. Assume that:

$$\text{(I-1)} \quad \begin{cases} \varphi_1 \in W^{1+\theta,p}(-r,0;D(A)), \quad \varphi_0 = \varphi_1(0)\ (\in D(A)), \\ f \in W^{1+\theta,p}(0,T;X) \cap C([0,T];(X,D(A))_{\theta+1-1/p,p}), \\ A\varphi_0 + A_1\varphi_1(-r) + \displaystyle\int_{-r}^0 a(s)A_2\varphi_1(s)ds \in (X,D(A))_{\theta+1-1/p,p}. \end{cases}$$

Theorem 1. *Suppose* $0 < \theta < 1/p$. *Then, under assumption* (I-1) *the solution of problem* (0.1) *satisfies*

$$u \in W^{1+\theta,p}(0,T;D(A)) \cap W^{2+\theta,p}(0,T;X) \cap C^1([0,T];(X,D(A))_{\theta+1-1/p,p}). \tag{1.1}$$

Since (I-1) is stronger than (I), in view of Theorem A a solution u of (0.1) exists and satisfies (0.2) and (0.3). Set

$$\varphi_0' = A\varphi_0 + A_1\varphi_1(-r) + \int_{-r}^0 a(s)A_2\varphi_1(s)ds + f(0). \tag{1.2}$$

Then (I-1) implies

$$\varphi_0' \in (X, D(A))_{\theta+1-1/p,p}, \;\; \varphi_1' \in W^{\theta,p}(-r, 0; D(A)), \;\; f' \in W^{\theta,p}(0, T; X).$$

Namely, (I) is satisfied by φ_0', φ_1', f' instead of φ_0, φ_1, f. Therefore there exists a unique solution v of the following problem:

$$\begin{cases} v'(t) = Av(t) + A_1 v(t-r) + \displaystyle\int_{-r}^{0} a(s) A_2 v(t+s) ds + f'(t), \;\; 0 \le t \le T, \\ v(0) = \varphi_0', \quad v(s) = \varphi_1'(s) \text{ a.e. } s \in (-r, 0) \end{cases} \tag{1.3}$$

satisfying

$$\begin{cases} v \in W^{\theta,p}(0, T; D(A)) \cap C([0, T]; (X, D(A))_{\theta+1-1/p,p}), \\ dv/dt \in W^{\theta,p}(0, T; X). \end{cases} \tag{1.4}$$

Set $w(t) = \varphi_0 + \int_0^t v(\tau) d\tau$. Then in view of (1.4)

$$\begin{cases} w' \in W^{\theta,p}(0, T; D(A)) \cap C([0, T]; (X, D(A))_{\theta+1-1/p,p}), \\ w'' \in W^{\theta,p}(0, T; X). \end{cases} \tag{1.5}$$

Since $\varphi_0 \in D(A)$, $v \in W^{\theta,p}(0, T; D(A)) \subset L^p(0, T; D(A))$, one has

$$w(\cdot) = \varphi_0 + \int_0^{\cdot} v(\tau) d\tau \in C([0, T]; D(A)) \subset L^p(0, T; D(A)). \tag{1.6}$$

In view of (1.5) $w' \in W^{\theta,p}(0, T; D(A)) \subset L^p(0, T; D(A))$. By virtue of this and (1.6)

$$w \in W^{1,p}(0, T; D(A)) \subset W^{\theta,p}(0, T; D(A)). \tag{1.7}$$

It follows from (1.7) and (1.5) that

$$w \in W^{1+\theta,p}(0, T; D(A)) \cap W^{2+\theta,p}(0, T; X).$$

Since $D(A) \subset (X, D(A))_{\theta+1-1/p,p}$, one has

$$w \in C([0, T]; D(A)) \subset C([0, T]; (X, D(A))_{\theta+1-1/p,p}).$$

From this and (1.5) it follows

$$w \in C^1([0, T]; (X, D(A))_{\theta+1-1/p,p}). \tag{1.8}$$

We are going to show

$$w(t) = u(t), \;\; 0 \le t \le T. \tag{1.9}$$

If this is proved, then (1.1) follows from (1.7) and (1.8).

Set $\breve{v}(t) = \begin{cases} v(t) & 0 \le t \le T, \\ \varphi_1'(t) & -r \le t \le 0. \end{cases}$ Then, $\breve{v} \in W^{\theta,p}(-r, T; D(A))$. Problem (1.3) is transformed to the following integral equation:

$$v(t) = e^{tA}\varphi_0' + \int_0^t e^{(t-s)A} A_1 \breve{v}(s-r) ds + \int_0^t e^{(t-s)A} \int_{-r}^{0} a(\sigma) A_2 \breve{v}(s+\sigma) d\sigma ds$$
$$+ \int_0^t e^{(t-s)A} f'(s) ds. \tag{1.10}$$

This implies

$$\int_0^t v(\tau)d\tau = \int_0^t e^{\tau A}\varphi_0' d\tau + \int_0^t \int_0^\tau e^{(\tau-s)A}A_1\breve{v}(s-r)dsd\tau$$
$$+\int_0^t \int_0^\tau e^{(\tau-s)A}\int_{-r}^0 a(\sigma)A_2\breve{v}(s+\sigma)d\sigma dsd\tau + \int_0^t \int_0^\tau e^{(\tau-s)A}f'(s)dsd\tau. \quad (1.11)$$

(i) Case $T \le r$. In view of the definition (1.2) of φ_0' one observes

$$\int_0^t e^{\tau A}\varphi_0' d\tau = [e^{tA}-I]A^{-1}\varphi_0'$$
$$= e^{tA}\varphi_0 - \varphi_0 + [e^{tA}-I]A^{-1}\left(A_1\varphi_1(-r) + \int_{-r}^0 a(s)A_2\varphi_1(s)ds + f(0)\right). \quad (1.12)$$

With the aid of a change of the order of integration and an integration by parts

$$\int_0^t \int_0^\tau e^{(\tau-s)A}A_1\breve{v}(s-r)dsd\tau = \int_0^t \int_0^\tau e^{(\tau-s)A}A_1\varphi_1'(s-r)dsd\tau$$
$$= \int_0^t [e^{(t-s)A}-I]A^{-1}A_1\varphi_1'(s-r)ds$$
$$= -[e^{tA}-I]A^{-1}A_1\varphi_1(-r) + \int_0^t e^{(t-s)A}A_1\varphi_1(s-r)ds. \quad (1.13)$$

Again changing the order of integration and integrating by parts one obtains

$$\int_0^t \int_0^\tau e^{(\tau-s)A}\int_{-r}^0 a(\sigma)A_2\breve{v}(s+\sigma)d\sigma dsd\tau = \int_0^t \int_s^t e^{(\tau-s)A}\int_{-r}^0 a(\sigma)A_2\breve{v}(s+\sigma)d\sigma d\tau ds$$
$$= \int_0^t\int_{-r}^0 a(\sigma)\int_s^t e^{(\tau-s)A}d\tau A_2\breve{v}(s+\sigma)d\sigma ds = \int_0^t\int_{-r}^0 a(\sigma)[e^{(t-s)A}-I]A^{-1}A_2\breve{v}(s+\sigma)d\sigma ds$$
$$= \int_0^t [e^{(t-s)A}-I]A^{-1}\int_{-r}^{-s} a(\sigma)A_2\varphi_1'(s+\sigma)d\sigma ds$$
$$+\int_0^t [e^{(t-s)A}-I]A^{-1}\int_{-s}^0 a(\sigma)A_2 v(s+\sigma)d\sigma ds = \mathrm{I}_1 + \mathrm{I}_2, \quad (1.14)$$

where

$$\mathrm{I}_1 = \int_0^t [e^{(t-s)A}-I]A^{-1}\int_{-r}^{-s} a(\sigma)A_2\varphi_1'(s+\sigma)d\sigma ds,$$
$$\mathrm{I}_2 = \int_0^t [e^{(t-s)A}-I]A^{-1}\int_{-s}^0 a(\sigma)A_2 v(s+\sigma)d\sigma ds.$$

Changing the order of integration and integrating by parts yield

$$
\begin{aligned}
\mathrm{I}_1 &= \int_{-r}^{-t} a(\sigma) \int_0^t [e^{(t-s)A} - I]A^{-1}A_2\varphi_1'(s+\sigma)dsd\sigma \\
&+ \int_{-t}^{0} a(\sigma) \int_0^{-\sigma} [e^{(t-s)A} - I]A^{-1}A_2\varphi_1'(s+\sigma)dsd\sigma \\
&= \int_{-r}^{-t} a(\sigma) \left\{ -[e^{tA} - I]A^{-1}A_2\varphi_1(\sigma) + \int_0^t e^{(t-s)A}A_2\varphi_1(s+\sigma)ds \right\} d\sigma \\
&+ \int_{-t}^{0} a(\sigma) \left\{ [e^{(t+\sigma)A} - I]A^{-1}A_2\varphi_1(0) - [e^{tA} - I]A^{-1}A_2\varphi_1(\sigma) \right\} d\sigma \\
&+ \int_{-t}^{0} a(\sigma) \left\{ \int_0^{-\sigma} e^{(t-s)A}A_2\varphi_1(s+\sigma)ds \right\} d\sigma \\
&= -\int_{-r}^{0} a(\sigma)[e^{tA} - I]A^{-1}A_2\varphi_1(\sigma)d\sigma + \int_{-r}^{-t} a(\sigma) \int_0^t e^{(t-s)A}A_2\varphi_1(s+\sigma)dsd\sigma \\
&+ \int_{-t}^{0} a(\sigma)[e^{(t+\sigma)A} - I]A^{-1}A_2\varphi_1(0)d\sigma + \int_{-t}^{0} a(\sigma) \int_0^{-\sigma} e^{(t-s)A}A_2\varphi_1(s+\sigma)dsd\sigma.
\end{aligned}
$$

The sum of the second and fourth terms of the last side of the above equalities is equal to

$$
\begin{aligned}
&\int_0^t e^{(t-s)A} \int_{-r}^{-t} a(\sigma)A_2\varphi_1(s+\sigma)d\sigma ds + \int_0^t e^{(t-s)A} \int_{-t}^{-s} a(\sigma)A_2\varphi_1(s+\sigma)d\sigma ds \\
&= \int_0^t e^{(t-s)A} \int_{-r}^{-s} a(\sigma)A_2\varphi_1(s+\sigma)d\sigma ds.
\end{aligned}
$$

Therefore

$$
\begin{aligned}
\mathrm{I}_1 &= -\int_{-r}^{0} a(\sigma)[e^{tA} - I]A^{-1}A_2\varphi_1(\sigma)d\sigma + \int_{-t}^{0} a(\sigma)[e^{(t+\sigma)A} - I]A^{-1}A_2\varphi_1(0)d\sigma \\
&+ \int_0^t e^{(t-s)A} \int_{-r}^{-s} a(\sigma)A_2\varphi_1(s+\sigma)d\sigma ds. \qquad (1.15)
\end{aligned}
$$

We can show without difficulty

$$
\begin{aligned}
\mathrm{I}_2 &= \int_{-t}^{0} a(\sigma) \int_{-\sigma}^{t} [e^{(t-s)A} - I]A^{-1}A_2 v(s+\sigma)dsd\sigma \\
&= \int_{-t}^{0} a(\sigma) \int_{-\sigma}^{t} [e^{(t-s)A} - I]A^{-1} \frac{\partial}{\partial s} \int_0^{s+\sigma} A_2 v(\tau)d\tau dsd\sigma \\
&= \int_{-t}^{0} a(\sigma) \int_{-\sigma}^{t} e^{(t-s)A} \int_0^{s+\sigma} A_2 v(\tau)d\tau dsd\sigma \\
&= \int_{-t}^{0} a(\sigma) \int_{-\sigma}^{t} e^{(t-s)A}A_2 \left(\varphi_0 + \int_0^{s+\sigma} v(\tau)d\tau \right) dsd\sigma - \int_{-t}^{0} a(\sigma) \int_{-\sigma}^{t} e^{(t-s)A}A_2\varphi_0 dsd\sigma \\
&= \int_{-t}^{0} a(\sigma) \int_{-\sigma}^{t} e^{(t-s)A}A_2 \left(\varphi_0 + \int_0^{s+\sigma} v(\tau)d\tau \right) dsd\sigma \\
&+ \int_{-t}^{0} a(\sigma)[I - e^{(t+\sigma)A}]A^{-1}A_2\varphi_0 d\sigma. \qquad (1.16)
\end{aligned}
$$

From (1.14) – (1.16) and $\varphi_0 = \varphi_1(0)$ it follows that

$$\int_0^t \int_0^\tau e^{(\tau-s)A} \int_{-r}^0 a(\sigma) A_2 v(s+\sigma) d\sigma ds d\tau = \mathrm{I}_1 + \mathrm{I}_2$$
$$= -\int_{-r}^0 a(\sigma)[e^{tA} - I]A^{-1}A_2\varphi_1(\sigma)d\sigma + \int_0^t e^{(t-s)A} \int_{-r}^{-s} a(\sigma)A_2\varphi_1(s+\sigma)d\sigma ds$$
$$+ \int_{-t}^0 a(\sigma) \int_{-\sigma}^t e^{(t-s)A} A_2 \left(\varphi_0 + \int_0^{s+\sigma} v(\tau)d\tau\right) ds d\sigma. \tag{1.17}$$

As is easily seen

$$\int_0^t \int_0^\tau e^{(\tau-s)A} f'(s) ds d\tau = \int_0^t \int_s^t e^{(\tau-s)A} d\tau f'(s) ds = \int_0^t [e^{(t-s)A} - I]A^{-1} f'(s) ds$$
$$= -[e^{tA} - I]A^{-1} f(0) + \int_0^t e^{(t-s)A} f(s) ds. \tag{1.18}$$

From (1.11) – (1.13), (1.17), (1.18) the following equality follows easily:

$$w(t) = e^{tA}\varphi_0 + \int_0^t e^{(t-s)A} A_1 \varphi_1(s-r) ds + \int_0^t e^{(t-s)A} \int_{-r}^{-s} a(\sigma) A_2 d\sigma ds$$
$$+ \int_{-t}^0 a(\sigma) \int_{-\sigma}^t e^{(t-s)A} A_2 w(s+\sigma) ds d\sigma + \int_0^t e^{(t-s)A} f(s) ds. \tag{1.19}$$

Set $\widehat{w}(t) = \begin{cases} w(t) & 0 \le t \le T, \\ \varphi_1(t) & -r \le t \le 0. \end{cases}$ Then (1.19) is rewritten as

$$w(t) = e^{tA}\varphi_0 + \int_0^t e^{(t-s)A} A_1 \widehat{w}(s-r) ds + \int_0^t e^{(t-s)A} \int_{-r}^0 a(\sigma) A_2 \widehat{w}(s+\sigma) d\sigma ds$$
$$+ \int_0^t e^{(t-s)A} f(s) ds.$$

Consequently (1.9) is obtained.

(ii) Case $r < T \le 2r$. By virtue of the result established in the previous case $0 < T \le r$ we already know that $w(t) = u(t)$ for $0 \le t \le r$. Hence

$$w(t) = \varphi_0 + \int_0^r v(\tau)d\tau + \int_r^t v(\tau)d\tau = w(r) + \int_r^t v(\tau)d\tau = u(r) + \int_r^t v(\tau)d\tau.$$

Since

$$Au(r) + A_1 u(0) + \int_{-r}^0 a(s) A_2 u(r+s) ds = u'(r) - f(r) \in (X, D(A))_{\theta+1-1/p,p},$$

the following facts hold:

$$\begin{cases} u_{[0,r]} \in W^{1+\theta,p}(0,r;D(A)), \\ f \in W^{1+\theta,p}(r,T;X) \cap C([r,T];(X,D(A))_{\theta+1-1/p,p}), \\ Au_{[0,r]}(r) + A_1 u_{[0,r]}(0) + \int_{-r}^0 a(s) A_2 u_{[0,r]}(r+s) ds \in (X, D(A))_{\theta+1-1/p,p}. \end{cases}$$

Hence (I-1) is satisfied with $[-r,0]$, φ_1 replaced by $[0,r]$, $u_{[0,r]}$ respectively. Therefore, by the method of the previous case we can show $w(t)=u(t)$ for $r\le t\le T$.

We can proceed to show (1.9) in the general case, and the proof of Theorem 1 is complete.

In case $1/p<\theta<1$ we assume

$$\text{(II-1)}\quad \begin{cases} \varphi_1\in W^{1+\theta,p}(-r,0;D(A)) \quad \big(\Longrightarrow \varphi_1(0)\in D(A)\big),\\ f\in W^{1+\theta,p}(0,T;X)\cap C^1[0,T];(X,D(A))_{\theta-1/p,p}),\\ A\varphi_1(0)+A_1\varphi_1(-r)+\displaystyle\int_{-r}^0 a(s)A_2\varphi_1(s)ds\in (X,D(A))_{\theta-1/p,p},\\ A\varphi_1'(0)+A_1\varphi_1'(-r)+\displaystyle\int_{-r}^0 a(s)A_2\varphi_1'(s)ds\in (X,D(A))_{\theta-1/p,p},\\ \varphi_1'(0)(=D_-\varphi_1(0))=A\varphi_1(0)+A_1\varphi_1(-r)+\int_{-r}^0 a(\sigma)A_2\varphi_1(\sigma)d\sigma+f(0). \end{cases}$$

Theorem 2. *Suppose* $1/p<\theta<1$. *If assumption* (II-1) *is satisfied, the solution* u *of* (0.1) *satisfies*

$$u\in W^{1+\theta,p}(0,T;D(A))\cap W^{2+\theta,p}(0,T;X)\cap C^2([0,T];(X,D(A))_{\theta-1/p,p}). \tag{1.20}$$

If hypothesis (II-1) holds, then (II) is satisfied by φ_1' and f' in place of φ_1 and f respectively. Therefore according to Theorem B there exists a unique solution v of the following problem

$$\frac{d}{dt}v(t)=Av(t)+A_1v(t-r)+\int_{-r}^0 a(s)A_2v(t+s)ds+f'(t), \tag{1.21}$$

$$v(s)=\varphi_1'(s),\ \ -r\le s\le 0 \tag{1.22}$$

satisfying

$$v\in W^{\theta,p}(0,T;D(A)), \tag{1.23}$$

$$dv/dt\in W^{\theta,p}(0,T;X)\cap C([0,T];(X,D(A))_{\theta-1/p}). \tag{1.24}$$

Set

$$w(t)=\varphi_0+\int_0^t v(\tau)d\tau. \tag{1.25}$$

In view of (1.23), (1.24) one has

$$w'\in W^{\theta,p}(0,T;D(A)), \tag{1.26}$$

$$w''\in W^{\theta,p}(0,T;X)\cap C([0,T];(X,D(A))_{\theta-1/p,p}). \tag{1.27}$$

Since $\theta>1/p$, $W^{\theta,p}(0,T;D(A))\subset C([0,T];D(A))$. Hence $v\in C([0,T];D(A))$. From this and $\varphi_0\in D(A)$ it follows that $w\in C^1([0,T];D(A))$. This implies $w\in W^{1,p}(0,T;D(A))\subset W^{\theta,p}(0,T;D(A))$. Hence with the aid of (1.26), (1.27) we deduce

$$w\in W^{1+\theta,p}(0,T;D(A))\cap W^{2+\theta,p}(0,T;X). \tag{1.28}$$

Since $D(A) \subset (X, D(A))_{\theta-1/p,p}$, one also has $w \in C^1([0,T]; (X, D(A))_{\theta-1/p,p})$. From this and (1.27) it follows that

$$w \in C^2([0,T]; (X, D(A))_{\theta-1/p,p}). \tag{1.29}$$

If it is shown that $w(t) = u(t)$, $0 \le t \le T$, then in view of (1.28) and (1.29) the proof of Theorem 2 is complete. This part of the proof is almost the same as that of Theorem 1, and so it is omitted.

2. Solution Semigroup: Case $\theta < 1/p$

Suppose assumption (I) is satisfied. Set

$$Z = (X, D(A))_{\theta+1-1/p,p} \times W^{\theta,p}(-r, 0; D(A)).$$

Following G. Di Blasio, K. Kunisch and E. Sinestrari [3] the solution semigroup for (0.1) is defined as follows:

$$S(t) \begin{pmatrix} \varphi_0 \\ \varphi_1 \end{pmatrix} = \begin{pmatrix} u(t) \\ \widehat{u}_t \end{pmatrix}, \ t \ge 0, \ \text{for} \begin{pmatrix} \varphi_0 \\ \varphi_1 \end{pmatrix} \in Z,$$

where u is the solution of problem (0.1) with $f(t) \equiv 0$, and $\widehat{u}(t) = \begin{cases} u(t) & 0 \le t < \infty, \\ \varphi_1(t) & -r \le t \le 0, \end{cases}$ $\widehat{u}_t(s) = \widehat{u}(t+s)$ for $-r \le s \le 0$. Since $u \in W^{\theta,p}(0,\infty; D(A)) \cap C([0,\infty); (X, D(A))_{\theta+1-1/p,p})$, where $u \in W^{\theta,p}(0,\infty; D(A))$ means $u \in W^{\theta,p}(0,T; D(A))$ for any $0 < T < \infty$, $u(t) \in (X, D(A))_{\theta+1-1/p,p}$ for $t \ge 0$, $\widehat{u} \in W^{\theta,p}(-r,\infty; D(A))$, and hence $\widehat{u}_t \in W^{\theta,p}(-r, 0; D(A))$ for $0 \le t < \infty$. Therefore $S(t) : Z \mapsto Z$ and $S(0) = I$. It can be shown without difficulty that $S(t)$ is a C_0-semigroup in Z.

Theorem 3. *The infinitesimal generater of the solution semigroup $S(t)$ is given by*

$$D(\Lambda) = \left\{ \begin{pmatrix} \varphi_0 \\ \varphi_1 \end{pmatrix}; \varphi_1 \in W^{1+\theta,p}(-r, 0; D(A)),\ \varphi_1(0) = \varphi_0\ , \right.$$
$$\left. A\varphi_0 + A_1\varphi_1(-r) + \int_{-r}^0 a(s) A_2 \varphi_1(s) ds \in (X, D(A))_{\theta+1-1/p,p} \right\},$$
$$\Lambda \begin{pmatrix} \varphi_0 \\ \varphi_1 \end{pmatrix} = \begin{pmatrix} A\varphi_0 + A_1\varphi_1(-r) + \int_{-r}^0 a(s) A_2 \varphi_1(s) ds \\ \varphi_1' \end{pmatrix}.$$

This theorem can be establised by showing the following statements following G. Di Blasio, K. Kunisch and E. Sinestrari [3]:

$$\begin{array}{l} \text{(i) } S(t)D(\Lambda) \subset D(\Lambda), \\ \text{(ii) } D(\Lambda) \text{ is dense in } Z, \\ \text{(iii) } \Lambda \subset \text{infinitesimal generator of } \{S(t)\}, \\ \text{(iv) } \Lambda : D(\Lambda) \subset Z \to Z \text{ is closed.} \end{array} \tag{2.1}$$

Problem (0.1) is rewritten as

$$\begin{cases} \dfrac{d}{dt} \begin{pmatrix} u(t) \\ \widehat{u}_t \end{pmatrix} = \Lambda \begin{pmatrix} u(t) \\ \widehat{u}_t \end{pmatrix} + \begin{pmatrix} f(t) \\ 0 \end{pmatrix}, \ 0 \le t \le T, \\ \begin{pmatrix} u(0) \\ \widehat{u}_0 \end{pmatrix} = \begin{pmatrix} \varphi_0 \\ \varphi_1 \end{pmatrix}. \end{cases} \tag{2.2}$$

The mild solution of (2.2) is expressed as

$$\begin{pmatrix} u(t) \\ \widehat{u}_t \end{pmatrix} = S(t) \begin{pmatrix} \varphi_0 \\ \varphi_1 \end{pmatrix} + \int_0^t S(t-s) \begin{pmatrix} f(s) \\ 0 \end{pmatrix} ds.$$

If $\begin{pmatrix} \varphi_0 \\ \varphi_1 \end{pmatrix} \in D(\Lambda)$ and $f \in C^1([0,T];(X,D(A))_{\theta+1-1/p,p})$, then $\begin{pmatrix} u(t) \\ \widehat{u}_t \end{pmatrix}$ is a strict solution, and

$$\begin{aligned} & u \in C^1([0,T];(X,D(A))_{\theta+1-1/p,p}), \\ & \widehat{u}_\cdot \in C^1([0,T];W^{\theta,p}(-r,0;D(A))), \\ & \frac{d}{dt}\begin{pmatrix} u(t) \\ \widehat{u}_t \end{pmatrix} = S(t)\Lambda \begin{pmatrix} \varphi_0 \\ \varphi_1 \end{pmatrix} + S(t)\begin{pmatrix} f(0) \\ 0 \end{pmatrix} + \int_0^t S(t-s)\begin{pmatrix} f'(s) \\ 0 \end{pmatrix} ds. \end{aligned} \tag{2.3}$$

Starting from

$$u_\cdot \in C([0,T];L^p(-r,0;D(A))) \Longleftrightarrow u \in L^p(-r,T;D(A))$$

one can show that (2.3) is equivalent to $\widehat{u} \in W^{1+\theta,p}(-r,T;D(A))$. Thus the following assertion holds:

Theorem 4. *If the following assumptions are satisfied*:

$$\begin{aligned} & \varphi_1 \in W^{1+\theta,p}(-r,0;D(A)), \quad \varphi_0 = \varphi_1(0), \quad f \in C^1([0,T];(X,D(A))_{\theta+1-1/p,p}), \\ & A\varphi_0 + A_1\varphi_1(-r) + \textstyle\int_{-r}^0 a(s)A_2\varphi_1(s)ds \in (X,D(A))_{\theta+1-1/p,p}, \end{aligned}$$

then a solution of (0.1) *satisfying*

$$u \in W^{1+\theta,p}(0,T;D(A)) \cap C^1([0,T];(X,D(A))_{\theta+1-1/p,p})$$

exists and is unique.

3. Regularity of Solutions: Case $\theta > 1/p$

In this section we suppose that the following assumptions are satisfied:

(II-2) $\quad A_1A^{-1}, A_2A^{-1} \in \mathcal{L}((X,D(A))_{\theta-1/p,p},(X,D(A))_{\theta-1/p,p})$,

(II-3) $\quad \begin{cases} \varphi_1 \in W^{\theta,p}(-r,0;D(A)) \cap C([-r,0];(X,D(A))_{\theta+1-1/p,p}), \\ f \in W^{\theta,p}(0,T;X) \cap C([0,T];(X,D(A))_{\theta-1/p,p}). \end{cases}$

Remark 1. Set $\varphi_0 = \varphi_1(0)$. Then it follows from (II-3) that $\varphi_0 \in (X,D(A))_{\theta+1-1/p,p}$. Hence $A\varphi_0 \in (X,D(A))_{\theta-1/p,p}$. From $\varphi_1 \in C([-r,0];(X,D(A))_{\theta+1-1/p,p})$ it follows that

$$A\varphi_1 \in C([-r,0];(X,D(A))_{\theta-1/p,p}), \quad \int_{-r}^0 a(s)A\varphi_1(s)ds \in (X,D(A))_{\theta-1/p,p}.$$

Hence by (II-2)

$$A_1\varphi_1 \in C([-r,0];(X,D(A))_{\theta-1/p,p}), \quad \int_{-r}^0 a(s)A_2\varphi_1(s)ds \in (X,D(A))_{\theta-1/p,p}. \tag{3.1}$$

Hence the final condition of (II) is satisfied. Therefore (II-2) and (II-3) imply (II).

Remark 2. (II-2) is equivalent to

$$A_i \in \mathcal{L}((X, D(A))_{\theta+1-1/p,p}, (X, D(A))_{\theta-1/p,p}),\ i = 1, 2.$$

A comment on assumption (II-2) will be given in the final section.

Theorem 5. *Suppose $\theta > 1/p$, and assumptions* (II-2) *and* (II-3) *are satisfied. Then the solution u of problem* (0.1) *satisfies*

$$u \in W^{\theta,p}(0, T; D(A)) \cap C([0, T]; (X, D(A))_{\theta+1-1/p,p}), \tag{3.2}$$

$$du/dt \in W^{\theta,p}(0, T; X) \cap C([0, T]; (X, D(A))_{\theta-1/p,p}). \tag{3.3}$$

Suppose first $T \le r$. Let u_0 be the function defined by

$$u_0(t) = e^{tA}[\varphi_1(0) + A^{-1}\widetilde{f}(0)] + \int_0^t e^{(t-s)A}\widetilde{f}_*(s)ds - A^{-1}\widetilde{f}(0), \tag{3.4}$$

where

$$\widetilde{f}(s) = A_1\varphi_1(s-r) + f(s) + \int_{-r}^0 a(\sigma)A_2\varphi_1(\sigma)d\sigma,$$

$$\widetilde{f}_*(s) = \widetilde{f}(s) - \widetilde{f}(0) = A_1\varphi_1(s-r) + f(s) - A_1\varphi_1(-r) - f(0).$$

It follows from (II-3) and (3.1) that

$$\widetilde{f} \in W^{\theta,p}(0, T; X) \cap C([0, T]; (X, D(A))_{\theta-1/p,p}), \tag{3.5}$$

$$\widetilde{f}_* \in W_*^{\theta,p}(0, T; X) \cap C([0, T]; (X, D(A))_{\theta-1/p,p}), \tag{3.6}$$

$$\varphi_0 + A^{-1}\widetilde{f}(0) \in (X, D(A))_{\theta+1-1/p,p}. \tag{3.7}$$

The solution of (0.1) is obtained as the solution of the following integral equation

$$u(t) = u_0(t) + \int_0^t e^{(t-s)A}\int_{-r}^0 a(\sigma)[A_2\widehat{u}(s+\sigma) - A_2\varphi_1(\sigma)]d\sigma ds, \tag{3.8}$$

where $\widehat{u}(s) = \begin{cases} u(s) & s \ge 0, \\ \varphi_1(s) & s < 0. \end{cases}$ This equation is sovled by successive approximation:

$$u_{n+1}(t) = u_0(t) + \int_0^t e^{(t-s)A}\int_{-r}^0 a(\sigma)[A_2\widehat{u}_n(s+\sigma) - A_2\varphi_1(\sigma)]d\sigma ds,\ n = 1, 2, 3, \ldots. \tag{3.9}$$

It is shown in A. Favini and H. Tanabe [2] that $u_n \in W^{\theta,p}(0, T; D(A))$, $u_n(0) = \varphi_0$, $n = 0, 1, 2, \ldots$. From (3.9) it follows that

$$u_{n+1}(t) - u_n(t) = \int_0^t e^{(t-s)A}\int_{-r}^0 a(\sigma)[A_2\widehat{u}_n(s+\sigma) - A_2\widehat{u}_{n-1}(s+\sigma)]d\sigma ds. \tag{3.10}$$

If $-r \le \sigma < -T$, then $s + \sigma < t - T \le 0$. Hence

$$A_2\widehat{u}_n(s+\sigma) - A_2\widehat{u}_{n-1}(s+\sigma) = \varphi_1(s+\sigma) - \varphi_1(s+\sigma) = 0.$$

Therefore (3.10) is rewritten as

$$u_{n+1}(t) - u_n(t) = \int_0^t e^{(t-s)A} \int_{-T}^0 a(\sigma)[A_2\widehat{u}_n(s+\sigma) - A_2\widehat{u}_{n-1}(s+\sigma)]d\sigma ds. \tag{3.11}$$

It is proved in [2] that

$$\|u_{n+1} - u_n\|_{W^{\theta,p}(0,T;D(A))} \le C_2' k_2 \|a\|_{L^1(-T,0)} \left[(\theta p)^{-1/p} C_0' + 1\right] \|u_n - u_{n-1}\|_{W^{\theta,p}(0,T;D(A))}$$

for some constants C_0', C_2' independent of T and $k_2 = \|A_2 A^{-1}\|$. Therefore if T is so small that

$$C_2' k_2 \|a\|_{L^1(-T,0)} \left[(\theta p)^{-1/p} C_0' + 1\right] < 1, \tag{3.12}$$

then

$$\sum_{n=1}^{\infty} \|u_{n+1} - u_n\|_{W^{\theta,p}(0,T;D(A))} < \infty. \tag{3.13}$$

Set

$$W_*^{\theta,p}(0,T;X) = \{u \in W^{\theta,p}(0,T;X);\ u(0) = 0\}.$$

The following lemma is due to G. Di Blasio [5] (Theorem 10 if $\theta < 1/p$ and Theorem 8 if $\theta > 1/p$). Also c.f. Lemma 1 of G. Di Blasio and A. Lorenzi [1].

Lemma 1. *Suppose $\theta \ne 1/p$. If $x \in (X, D(A))_{\theta+1-1/p,p}$, then*

$$e^{-\cdot A} x \in W^{\theta,p}(0,T;D(A)) \cap C([0,T];(X,D(A))_{\theta+1-1/p,p}).$$

The following lemma is Theorem 24 of G. Di Blasio [5].

Lemma 2. *Suppose $\theta > 1/p$. Then, if $f \in W_*^{\theta,p}(0,T;X)$, the function $V(t) = \int_0^t e^{(t-s)A} f(s) ds$ satisfies*

$$\begin{aligned} &V \in W^{\theta,p}(0,T;D(A)), \\ &dV/dt = AV + f \in C([0,T];(X,D(A))_{\theta-1/p,p}), \end{aligned} \tag{3.14}$$

and the following inequality holds with a constant C_2' independent of T:

$$\|V\|_{W^{\theta,p}(0,T;D(A))} \le C_2' \|f\|_{W^{\theta,p}(0,T;X)}. \tag{3.15}$$

In A. Favini and H. Tanabe [2] it is shown that the constant C_2' above can be chosen independent of T if we choose (0.6) as the norm of $W^{\theta,p}(0,T;D(A))$.

Let V and f be as in Lemma 2. With the aid of (3.14) and (3.15) one observes

$$\begin{aligned} \|V'\|_{W^{\theta,p}(0,T;X)} &= \|AV + f\|_{W^{\theta,p}(0,T;X)} \le \|V\|_{W^{\theta,p}(0,T;D(A))} + \|f\|_{W^{\theta,p}(0,T;X)} \\ &\le (C_2' + 1)\|f\|_{W^{\theta,p}(0,T;X)}. \end{aligned} \tag{3.16}$$

From Lemma 2 the following lemma follows:

Lemma 3. *Suppose $\theta > 1/p$. Let $f \in W_*^{\theta,p}(0,T;X) \cap C([0,T];(X,D(A))_{\theta-1/p,p})$. Then, for $V(t) = \int_0^t e^{(t-s)A} f(s) ds$ one has*

$$V \in W^{\theta,p}(0,T;D(A)) \cap C([0,T];(X,D(A))_{\theta+1-1/p,p}).$$

In view of Lemma 1, 3 and (3.6), (3.7) one observes

$$e^{-\cdot A}[\varphi_1(0) + A^{-1}\widetilde{f}(0)] \in C([0,T];(X,D(A))_{\theta+1-1/p,p}),$$
$$\textstyle\int_0^{\cdot} e^{-(\cdot-s)A}\widetilde{f}_*(s)ds \in C([0,T];(X,D(A))_{\theta+1-1/p,p}).$$

Moreover, $A^{-1}\widetilde{f}(0)$ is a constant function with a value in $(X,D(A))_{\theta+1-1/p,p}$. Consequently

$$u_0 \in C([0,T];(X,D(A))_{\theta+1-1/p,p}). \tag{3.17}$$

Suppose for some $n = 1, 2, \dots$

$$u_n \in C([0,T];(X,D(A))_{\theta+1-1/p,p}). \tag{3.18}$$

Then $\widehat{u}_n \in C([-r,T];(X,D(A))_{\theta+1-1/p,p})$. Hence $A_2\widehat{u}_n \in C([-r,T];(X,D(A))_{\theta-1/p,p})$ in view of Remark 2. Therefore it is easy to show

$$\int_{-r}^{0} a(\sigma)[A_2\widehat{u}_n(\cdot+\sigma) - A_2\varphi_1(\sigma)]d\sigma \in C([0,T];(X,D(A))_{\theta-1/p,p}). \tag{3.19}$$

Since $u_n \in W^{\theta,p}(0,T;D(A))$, one has $\widehat{u}_n \in W^{\theta,p}(-r,T;D(A))$, and hence $A_2\widehat{u}_n \in W^{\theta,p}(-r,T;X)$. The following lemma is proved in A. Favini and H. Tanabe [2, Lemma 2.5]:

Lemma 4. *Suppose $v \in W^{\theta,p}(-r,T;X)$, $0 < \theta < 1$, $\theta \neq 1/p$. Then $\int_{-r}^{0} a(\sigma)v(\cdot+\sigma)d\sigma \in W^{\theta,p}(0,T;X)$, and*

$$\left\| \int_{-r}^{0} a(\sigma)v(\cdot+\sigma)d\sigma \right\|_{W^{\theta,p}(0,T;X)} \leq \|a\|_{L^1(-r,0)}\left[N_{\theta,p,(-r,T)}(v) + T^{-\theta}\|v\|_{L^p(-r,T;X)}\right].$$

Applying Lemma 4 to $A_2\widehat{u}_n$ one observes $\int_{-r}^{0} a(\sigma)A_2\widehat{u}_n(\cdot+\sigma)d\sigma \in W^{\theta,p}(0,T;X)$. Therefore, noting (3.1) one deduces

$$\begin{aligned}&\int_{-r}^{0} a(\sigma)[A_2\widehat{u}_n(\cdot+\sigma) - A_2\varphi_1(\sigma)]d\sigma \\ &= \int_{-r}^{0} a(\sigma)A_2\widehat{u}_n(\cdot+\sigma)d\sigma - \int_{-r}^{0} a(\sigma)A_2\varphi_1(\sigma)d\sigma \in W_*^{\theta,p}(0,T;X).\end{aligned} \tag{3.20}$$

By virtue of (3.19), (3.20) and Lemma 3 one obtains

$$\int_0^{\cdot} e^{(\cdot-s)A}\int_{-r}^{0} a(\sigma)[A_2\widehat{u}_n(s+\sigma) - A_2\varphi_1(\sigma)]d\sigma ds \in C([0,T];(X,D(A))_{\theta+1-1/p,p}). \tag{3.21}$$

From (3.9), (3.17) and (3.21) it follows that (3.18) holds with $n+1$ in place of n.

Next, we estimate the following norm:

$$\|V'\|_{L^p(0,T;(X,D(A))_{\theta,p})} = \left(\|V'\|^p_{L^p(0,T;X)} + \int_0^T |V'(t)|^p_{\theta,p}dt\right)^{1/p}, \tag{3.22}$$

where $|\cdot|_{\theta,p}$ is the semi norm defined by

$$|u|_{\theta,p} = \left(\int_0^\infty \|t^{1-\theta} A e^{tA} u\|^p t^{-1} dt \right)^{1/p}.$$

The following inequality was shown in the proof of Lemma 2 of G. Di Blasio [5]:

$$\|V'\|_{L^p(0,T;X)} \le ((p-1)/\theta p)^{(p-1)/p} M_1 T^\theta N_{\theta,p,(0,T)}(f) + M_0 \|f\|_{L^p(0,T;X)}, \tag{3.23}$$

where M_0, M_1 are constants such that $\|e^{tA}\| \le M_0$, $\|(d/dt)e^{tA}\| \le M_1/t$. In the proof of Theorem 26 of [5, p. 81] it was shown that

$$\int_0^T |V'(t)|_{\theta,p}^p dt \le 2^{4p-2} M_1^{2p} (\theta^{-p} + (1-\theta)^{-p}) \left(N_{\theta,p,(0,T)}^p(f) + (\theta p)^{-1} \int_0^T t^{-\theta p} \|f(t)\|^p dt \right)$$
$$+ 2^{p-1} M_1^p \int_0^T t^{-\theta p} \|f(t)\|^p dt \int_0^\infty (s+1)^{-p} s^{-1+p-p\theta} ds. \tag{3.24}$$

It is easy to show the following inequality holds for $f \in W_*^{\theta,p}(0,T;X)$ with a constant c independent of T:

$$\int_0^T t^{-\theta p} \|f(t)\|^p dt \le c \|f\|_{W^{\theta,p}(0,T;X)}^p. \tag{3.25}$$

From (3.22) – (3.24) and (3.25) it follows that the following inequality holds with a constant C_1 independent of T:

$$\int_0^T |V'(t)|_{\theta,p}^p dt \le C_1 \|f\|_{W^{\theta,p}(0,T;X)}^p. \tag{3.26}$$

By virtue of (3.22), (3.23) and (3.26) the following inequality holds for $f \in W_*^{\theta,p}(0,T;X)$ with a constant C_2 independent of T:

$$\|V'\|_{L^p(0,T;(X,D(A))_{\theta,p})} \le C_2 (T^\theta + 1) \|f\|_{W^{\theta,p}(0,T;X)}. \tag{3.27}$$

The following lemma is also due to Lemma 11 of G. di Blasio [5].

Lemma 5. *Suppose $\theta > p$. Then*

$$W^{\theta,p}(0,T;X) \cap L^p(0,T;(X,D(A))_{\theta,p}) \subset C([0,T];(X,D(A))_{\theta-1/p,p}),$$

and the following inequality holds for $u \in W^{\theta,p}(0,T;X) \cap L^p(0,T;(X,D(A))_{\theta,p})$ with a constant C_3 independent of T:

$$\|u\|_{C([0,T];(X,D(A))_{\theta-1/p,p})} \le C_3 \left(T^{\theta-1/p} \|u\|_{W^{\theta,p}(0,T;X)} + T^{-1/p} \|u\|_{L^p(0,T;(X,D(A))_{\theta,p})} \right). \tag{3.28}$$

Inequality (3.28) follows from the one in case $T = 1$ and considering a function $u(Tt), 0 \le t \le 1$, in the general case.

In view of Lemma 5

$$\begin{aligned} &\|V'\|_{C([0,T];(X,D(A))_{\theta-1/p,p})} \le \\ &\le C_3 \left(T^{\theta-1/p} \|V'\|_{W^{\theta,p}(0,T;X)} + T^{-1/p} \|V'\|_{L^p(0,T;(X,D(A))_{\theta,p})} \right). \end{aligned} \tag{3.29}$$

Inequalities (3.16), (3.27) and (3.29) yield

$$\|V'\|_{C([0,T];(X,D(A))_{\theta-1/p,p})} \leq C_T \|f\|_{W^{\theta,p}(0,T;X)}, \tag{3.30}$$

where $C_T = C_3 \left((C_2 + C_2' + 1)T^\theta + C_2 \right) T^{-1/p}$. Hence

$$\begin{aligned} &\|V\|_{C([0,T];(X,D(A))_{\theta+1-1/p,p})} = \|AV\|_{C([0,T];(X,D(A))_{\theta-1/p,p})} \\ &= \|V' - f\|_{C([0,T];(X,D(A))_{\theta-1/p,p})} \leq C_T \|f\|_{W^{\theta,p}(0,T;X)} + \|f\|_{C([0,T];(X,D(A))_{\theta-1/p,p})}. \end{aligned} \tag{3.31}$$

We apply (3.31) to $\int_{-T}^0 a(\sigma)[A_2\widehat{u}_n(\cdot+\sigma) - A_2\widehat{u}_{n-1}(\cdot+\sigma)]d\sigma$. Then $V = u_{n+1} - u_n$ (c.f. (3.11)). Let T satisfy (3.12):

$$C_2' k_2 \|a\|_{L^1(-T,0)} \left[(\theta p)^{-1/p} C_0' + 1 \right] < 1.$$

One has

$$\begin{aligned} &\|u_{n+1} - u_n\|_{C([0,T];(X,D(A))_{\theta+1-1/p,p})} \\ &\leq C_T \left\| \int_{-T}^0 a(\sigma)[A_2\widehat{u}_n(\cdot+\sigma) - A_2\widehat{u}_{n-1}(\cdot+\sigma)]d\sigma \right\|_{W^{\theta,p}(0,T;X)} \\ &+ \left\| \int_{-T}^0 a(\sigma)[A_2\widehat{u}_n(\cdot+\sigma) - A_2\widehat{u}_{n-1}(\cdot+\sigma)]d\sigma \right\|_{C([0,T];(X,D(A))_{\theta-1/p,p})}. \end{aligned} \tag{3.32}$$

It is shown in [2] that the following inequality holds:

$$\begin{aligned} &\left\| \int_{-T}^0 a(\sigma)[A_2\widehat{u}_n(\cdot+\sigma) - A_2\widehat{u}_{n-1}(\cdot+\sigma)]d\sigma \right\|_{W^{\theta,p}(0,T;X)} \\ &\leq k_2 \|a\|_{L^1(-T,0)} \left[(\theta p)^{-1/p} C_0' + 1 \right] \|u_n - u_{n-1}\|_{W^{\theta,p}(0,T;D(A))}. \end{aligned} \tag{3.33}$$

As is easily seen

$$\begin{aligned} &\left\| \int_{-T}^0 a(\sigma)[A_2\widehat{u}_n(\cdot+\sigma) - A_2\widehat{u}_{n-1}(\cdot+\sigma)]d\sigma \right\|_{C([0,T];(X,D(A))_{\theta-1/p,p})} \\ &= \sup_{s\in[0,T]} \left\| \int_{-T}^0 a(\sigma)[A_2\widehat{u}_n(s+\sigma) - A_2\widehat{u}_{n-1}(s+\sigma)]d\sigma \right\|_{(X,D(A))_{\theta-1/p,p}} \\ &\leq \int_{-T}^0 |a(\sigma)|d\sigma \sup_{\tau\in[-T,T]} \|A_2\widehat{u}_n(\tau) - A_2\widehat{u}_{n-1}(\tau)\|_{(X,D(A))_{\theta-1/p,p}} \\ &\leq k_2 \|a\|_{L^1(-T,0)} \sup_{\tau\in[0,T]} \|Au_n(\tau) - Au_{n-1}(\tau)\|_{(X,D(A))_{\theta-1/p,p}} \\ &= k_2 \|a\|_{L^1(-T,0)} \|u_n - u_{n-1}\|_{C([0,T];(X,D(A))_{\theta+1-1/p,p})}. \end{aligned} \tag{3.34}$$

The following inequality follows from (3.32) – (3.34):

$$\begin{aligned} &\|u_{n+1} - u_n\|_{C([0,T];(X,D(A))_{\theta+1-1/p,p})} \\ &\leq C_T k_2 \|a\|_{L^1(-T,0)} \left[(\theta p)^{-1/p} C_0' + 1 \right] \|u_n - u_{n-1}\|_{W^{\theta,p}(0,T;D(A))} \\ &+ k_2 \|a\|_{L^1(-T,0)} \|u_n - u_{n-1}\|_{C([0,T];(X,D(A))_{\theta+1-1/p,p})}. \end{aligned}$$

Summing both sides from $n = 1$ to m one gets

$$\sum_{n=1}^{m} \|u_{n+1} - u_n\|_{C([0,T];(X,D(A))_{\theta+1-1/p,p})}$$
$$\leq C_T k_2 \|a\|_{L^1(-T,0)} \left[(\theta p)^{-1/p} C_0' + 1\right] \sum_{n=1}^{m} \|u_n - u_{n-1}\|_{W^{\theta,p}(0,T;D(A))}$$
$$+ k_2 \|a\|_{L^1(-T,0)} \sum_{n=1}^{m} \|u_n - u_{n-1}\|_{C([0,T];(X,D(A))_{\theta+1-1/p,p})}. \tag{3.35}$$

Substituting

$$\sum_{n=1}^{m} \|u_n - u_{n-1}\|_{C([0,T];(X,D(A))_{\theta+1-1/p,p})}$$
$$\leq \sum_{n=1}^{m} \|u_{n+1} - u_n\|_{C([0,T];(X,D(A))_{\theta+1-1/p,p})} + \|u_1 - u_0\|_{C([0,T];(X,D(A))_{\theta+1-1/p,p})}$$

in the last side of (3.35) one gets

$$\sum_{n=1}^{m} \|u_{n+1} - u_n\|_{C([0,T];(X,D(A))_{\theta+1-1/p,p})}$$
$$\leq C_T k_2 \|a\|_{L^1(-T,0)} \left[(\theta p)^{-1/p} C_0' + 1\right] \sum_{n=1}^{m} \|u_n - u_{n-1}\|_{W^{\theta,p}(0,T;D(A))}$$
$$+ k_2 \|a\|_{L^1(-T,0)} \sum_{n=1}^{m} \|u_{n+1} - u_n\|_{C([0,T];(X,D(A))_{\theta+1-1/p,p})}$$
$$+ k_2 \|a\|_{L^1(-T,0)} \|u_1 - u_0\|_{C([0,T];(X,D(A))_{\theta+1-1/p,p})}.$$

This implies

$$\left(1 - k_2 \|a\|_{L^1(-T,0)}\right) \sum_{n=1}^{m} \|u_{n+1} - u_n\|_{C([0,T];(X,D(A))_{\theta+1-1/p,p})}$$
$$\leq C_T k_2 \|a\|_{L^1(-T,0)} \left[(\theta p)^{-1/p} C_0' + 1\right] \sum_{n=1}^{m} \|u_n - u_{n-1}\|_{W^{\theta,p}(0,T;D(A))}$$
$$+ k_2 \|a\|_{L^1(-T,0)} \|u_1 - u_0\|_{C([0,T];(X,D(A))_{\theta+1-1/p,p})}. \tag{3.36}$$

Letting $m \to \infty$ in (3.36) one obtains in view of (3.13)

$$\left(1 - k_2 \|a\|_{L^1(-T,0)}\right) \sum_{n=1}^{\infty} \|u_{n+1} - u_n\|_{C([0,T];(X,D(A))_{\theta+1-1/p,p})}$$
$$\leq C_T k_2 \|a\|_{L^1(-T,0)} \left[(\theta p)^{-1/p} C_0' + 1\right] \sum_{n=1}^{\infty} \|u_n - u_{n-1}\|_{W^{\theta,p}(0,T;D(A))}$$
$$+ k_2 \|a\|_{L^1(-T,0)} \|u_1 - u_0\|_{C([0,T];(X,D(A))_{\theta+1-1/p,p})} < \infty. \tag{3.37}$$

Let T satisfy $k_2\|a\|_{L^1(-T,0)} < 1$ besides $T \le r$ and (3.12). Then by virtue of (3.37) and (3.13) one obtains

$$\sum_{n=1}^{\infty} \|u_{n+1} - u_n\|_{C([0,T];(X,D(A))_{\theta+1-1/p,p})} < \infty.$$

Hence $\{u_n\}$ is convergent in $C([0,T];(X,D(A))_{\theta+1-1/p,p})$. Since $u_n \to u$ in $W^{\theta,p}(0,T;D(A))$, one concludes $u \in C([0,T];(X,D(A))_{\theta+1-1/p,p})$ and

$$u_n \to u \text{ in } W^{\theta,p}(0,T;D(A)) \cap C([0,T];(X,D(A))_{\theta+1-1/p,p}).$$

Let T_0 satisfy

$$0 < T_0 \le r, \ \ C_2' k_2 \|a\|_{L^1(-T_0,0)} \left[(\theta p)^{-1/p} C_0' + 1\right] < 1, \ \ k_2\|a\|_{L^1(-T_0,0)} < 1.$$

Then, by the result just proved one has $u \in W^{\theta,p}(0,T;D(A)) \cap C([0,T_0];(X,D(A))_{\theta+1-1/p,p})$. Hence

$$\widehat{u} \in W^{\theta,p}(-r,T;D(A)) \cap C([-r,T_0];(X,D(A))_{\theta+1-1/p,p}).$$

Suppose $T_0 < T$. Then

$$\widehat{u}|_{[T_0-r,T_0]} \in W^{\theta,p}(T_0-r,T_0;D(A)) \cap C([T_0-r,T_0];(X,D(A))_{\theta+1-1/p,p}),$$

and u satisfies

$$\begin{cases} \dfrac{d}{dt}u(t) = Au(t) + A_1u(t-r) + \displaystyle\int_{-r}^{0} a(s)A_2u(t+s)ds + f(t), \ \ T_0 \le t \le T, \\ u(s) = \widehat{u}|_{[T_0-r,T_0]}(s), \ \ T_0 - r \le s \le T_0. \end{cases}$$

Therefore, by virtue of the result already proved

$$u \in W^{\theta,p}(0,T;D(A)) \cap C([0,\min\{2T_0,T\}];(X,D(A))_{\theta+1-1/p,p}).$$

Continuing this process we can complete the proof of Theorem 5.

Next, we consider the case where the following assumption is satisfied:

$$\text{(II-4)} \quad \begin{cases} \varphi_1 \in W^{1+\theta,p}(-r,0;D(A)) \cap C^1([-r,0];(X,D(A))_{\theta+1-1/p,p}), \\ f \in W^{\theta+1,p}(0,T;X) \cap C([0,T];(X,D(A))_{\theta+1-1/p,p}) \cap C^1([0,T];(X,D(A))_{\theta-1/p,p}), \\ \varphi_1'(0) = A\varphi_1(0) + A_1\varphi_1(-r) + \int_{-r}^{0} a(\sigma)A_2\varphi_1(\sigma)d\sigma + f(0). \end{cases}$$

Theorem 6. *If assumptions* (II-2) *and* (II-4) *are satisfied, then the solution u of* (0.1) *satisfies*

$$\begin{aligned} &u \in W^{1+\theta,p}(0,T;D(A)) \cap W^{2+\theta,p}(0,T;X) \cap C^1([0,T];(X,D(A))_{\theta+1-1/p,p}), \\ &u'' \in C([0,T];(X,D(A))_{\theta-1/p}). \end{aligned} \tag{3.38}$$

Proof. If hypotheses (II-2) and (II-4) are satisfied, then (II-1) and (II-3) are also satisfied. In view of Theorem 2 and Theorem 5 it suffices to show

$$u' \in C([0,T];(X,D(A))_{\theta+1-1/p}).$$

Since (II-4) is satisfied, (II-3) is satisfied by φ_1', f' in place of φ_1, f. Therefore in view of Theorem 5 there exists a solution v of the following problem

$$\frac{d}{dt}v(t) = Av(t) + A_1 v(t-r) + \int_{-r}^{0} a(s)A_2 v(t+s)ds + f'(t),$$
$$v(s) = \varphi_1'(s), \ -r \le s \le 0$$

satisfying

$$v \in W^{\theta,p}(0,T;D(A)) \cap C([0,T];(X,D(A))_{\theta+1-1/p,p}),$$
$$dv/dt \in W^{\theta,p}(0,T;X) \cap C([0,T];(X,D(A))_{\theta-1/p}).$$

Since $u(t) = \varphi_1(0) + \displaystyle\int_0^t v(\tau)d\tau$ (c.f. Proof of Theorem 2),

$$u' = v \in C([0,T];(X,D(A))_{\theta+1-1/p,p}).$$

□

4. Solution Semigroup: Case $\theta > 1/p$

In this section we assume that hypotheses (II-2) and (II-3) are satisfied. Let

$$Z = W^{\theta,p}(-r,0;D(A)) \cap C([-r,0];(X,D(A))_{\theta+1-1/p,p}).$$

For $\varphi_1 \in Z$ let u be the solution of (0.1) with $f(t) = 0$:

$$\frac{d}{dt}u(t) = Au(t) + A_1 u(t-r) + \int_{-r}^{0} a(s)A_2 u(t+s)ds, \ \ 0 \le t < \infty,$$
$$u(s) = \varphi_1(s), \ \ -r \le s \le 0. \tag{4.1}$$

In view of Theorem 5 u satisfies

$$u \in W^{\theta,p}(0,T;D(A)) \cap C([0,T];(X,D(A))_{\theta+1-1/p,p}),$$
$$u' \in W^{\theta,p}(0,T;X) \cap C([0,T];(X,D(A))_{\theta-1/p,p}).$$

Therefore $\widehat{u} \in W^{\theta,p}(-r,\infty;D(A)) \cap C([-r,\infty);(X,D(A))_{\theta+1-1/p,p})$, where $u \in W^{\theta,p}(-r,\infty;D(A))$ means $u \in W^{\theta,p}(-r,T;D(A))$ $\forall T > 0$. This implies $\widehat{u}_t \in Z$ for $t \ge 0$. Therefore if we set

$$S(t)\varphi_1 = \widehat{u}_t, \ \ t \ge 0,$$

$S(t)$ maps Z to Z.

Let $\widehat{u}$ be the solution of (4.1), v be the solution of (4.1) with the initial function $\widehat{u}_\tau$ and $w(t) = \widehat{u}(t+\tau)$ for $\tau > 0$, $t > 0$. Then

$$\frac{d}{dt}v(t) = Av(t) + A_1 v(t-r) + \int_{-r}^{0} a(s)v(t+s)ds, \ \ 0 \le t < \infty,$$
$$v(s) = \widehat{u}(\tau+s), \ \ -r \le s \le 0,$$
$$\frac{d}{dt}w(t) = \frac{d}{dt}\widehat{u}(t+\tau) = A\widehat{u}(t+\tau) + A_1\widehat{u}(t+\tau-r) + \int_{-r}^{0} a(s)\widehat{u}(t+\tau+s)ds$$
$$= Aw(t) + A_1 w(t-r) + \int_{-r}^{0} a(s)w(t+s)ds, \ \ 0 \le t < \infty,$$
$$w(s) = \widehat{u}(\tau+s), \ \ -r \le s \le 0.$$

Therefore $v = w$, and hence $v_t = w_t$. On the other hand

$$v_t = S(t)\widehat{u}_\tau = S(t)S(\tau)\varphi_1,$$
$$w_t(s) = w(t+s) = \widehat{u}(t+s+\tau) = \widehat{u}_{t+\tau}(s) = (S(t+\tau)\varphi_1)(s).$$

Thus

$$S(t)S(\tau)\varphi_1 = S(t+\tau)\varphi_1.$$

It is easy to see that the mapping $[0,T] \ni t \mapsto \widehat{u}_t \in C([-r,0];(X,D(A))_{\theta+1-1/p,p})$ is continuous. The continuity of $[0,T] \ni t \mapsto \widehat{u}_t \in W^{\theta,p}(-r,0;D(A))$ is shown in the following lemma.

Lemma 6. *For $v \in W^{\theta,p}(-r,T;D(A))$ the mapping $[0,T] \ni t \mapsto v_t \in W^{\theta,p}(-r,0;D(A))$ is continuous.*

Proof. The lemma is proved by the following step:
(i) For $w \in W^{1,p}(-r,T;D(A))$ $\lim_{\tau\to t} \|w_\tau - w_t\|_{W^{\theta,p}(-r,0;D(A))} = 0$.
(ii) For $v \in W^{\theta,p}(-r,T;D(A))$, $w \in W^{1,p}(-r,T;D(A))$ such that $\|v-w\|_{W^{\theta,p}(-r,T;D(A))} < \epsilon$ one has

$$\begin{aligned}
&\|v_\tau - v_t\|_{W^{\theta,p}(-r,0;D(A))}\\
&\le \|w_\tau - w_t\|_{W^{\theta,p}(-r,0;D(A))} + \|v_\tau - w_\tau\|_{W^{\theta,p}(-r,0;D(A))} + \|v_t - w_t\|_{W^{\theta,p}(-r,0;D(A))}\\
&\le \|w_\tau - w_t\|_{W^{\theta,p}(-r,0;D(A))} + 2\|v-w\|_{W^{\theta,p}(-r,T;D(A))} < \|w_\tau - w_t\|_{W^{\theta,p}(-r,0;D(A))} + 2\epsilon.
\end{aligned}$$

□

Hence the mapping $[0,T] \ni t \mapsto S(t)\varphi_1 = \widehat{u}_t \in Z$ is continuous. Thus it has been shown that $\{S(t),\, t \ge 0\}$ is a C_0-semigroup.

The following result is an analog to Theorem 4.4 of E. Sinestrari [4]:

Theorem 7. *The infinitesimal generator of $S(t)$ is given by*

$$\begin{aligned}
D(\Lambda) = \Big\{&\varphi_1 \in W^{1+\theta,p}(-r,0;D(A)) \cap C^1([-r,0];(X,D(A))_{\theta+1-1/p,p});\\
&\varphi_1'(0) = A\varphi_1(0) + A_1\varphi_1(-r) + \int_{-r}^0 a(\sigma)A_2\varphi_1(\sigma)d\sigma\Big\},\\
\Lambda\varphi_1 = \varphi_1'.&
\end{aligned}$$

Proof. We show

(i) $S(t)D(\Lambda) \subset D(\Lambda)$.

(ii) $D(\Lambda)$ is dense in $Z = W^{\theta,p}(-r,0;D(A)) \cap C([-r,0];(X,D(A))_{\theta+1-1/p,p})$.

(iii) $\Lambda \subset$ infinitesimal generator of $\{S(t)\}$.

(iv) $\Lambda : D(\Lambda) \subset W^{\theta,p}(-r,0;D(A)) \to W^{\theta,p}(-r,0;D(A))$ is closed.

(i) Let $\varphi_1 \in D(\Lambda)$ and u be the solution of (4.1). Then in view of Theorem 6 and its proof $u(0) = \varphi_1(0)$ and $u'(0) = \varphi_1'(0)$. Hence $\widehat{u} \in W^{1+\theta,p}(-r,\infty;D(A)) \cap C^1([-r,\infty];(X,D(A))_{\theta+1-1/p,p})$. Hence

$$\widehat{u}_t \in W^{1+\theta,p}(-r,0;D(A)) \cap C^1([-r,0];(X,D(A))_{\theta+1-1/p,p}).$$

For $t > 0$

$$
\begin{aligned}
(\widehat{u}_t)'(0) &= \lim_{s\to 0}\frac{\widehat{u}_t(s)-\widehat{u}_t(0)}{s} = \lim_{s\to -0}\frac{\widehat{u}(t+s)-\widehat{u}(t)}{s} = \lim_{s\to -0}\frac{u(t+s)-u(t)}{s} \\
&= u'(t) = Au(t) + A_1\widehat{u}(t-r) + \int_{-r}^{0} a(s)A_2\widehat{u}(t+s)ds \\
&= A\widehat{u}_t(0) + A_1\widehat{u}_t(-r) + \int_{-r}^{0} a(s)A_2\widehat{u}_t(s)ds.
\end{aligned}
$$

Hence $\widehat{u}_t \in D(\Lambda)$.

(ii) Let $\varphi_1 \in Z$ and $\widehat{u}_t = S(t)\varphi_1$. Set $\varphi_\epsilon = \int_0^\epsilon \widehat{u}_t dt$. $\varphi_\epsilon(s) = \int_0^\epsilon \widehat{u}_t(s)dt = \int_0^\epsilon \widehat{u}(t+s)dt$.

$$
\begin{aligned}
\varphi_\epsilon'(s) &= \frac{d}{ds}\left(\int_0^\epsilon \widehat{u}_t dt\right)(s) = \frac{d}{ds}\int_0^\epsilon \widehat{u}(t+s)dt = \int_0^\epsilon \widehat{u}'(t+s)dt \\
&= \widehat{u}(\epsilon+s) - \widehat{u}(s) = \widehat{u}(\epsilon+s) - \varphi_1(s) = \widehat{u}_\epsilon(s) - \varphi_1(s),\ -r \le s \le 0.
\end{aligned} \tag{4.2}
$$

Therefore

$$
\varphi_\epsilon' = \widehat{u}_\epsilon - \varphi_1 \in W^{\theta,p}(-r,0;D(A)) \cap C([-r,0];(X,D(A))_{\theta+1-1/p,p}).
$$

Hence

$$
\varphi_\epsilon \in W^{\theta+1,p}(-r,0;D(A)) \cap C^1([-r,0];(X,D(A))_{\theta+1-1/p,p}).
$$

By virtue of (4.2)

$$
\begin{aligned}
&A\varphi_\epsilon(0) + A_1\varphi_\epsilon(-r) + \int_{-r}^{0} a(\sigma)A_2\varphi_\epsilon(\sigma)d\sigma \\
&= A\int_0^\epsilon \widehat{u}(t)dt + A_1\int_0^\epsilon \widehat{u}(t-r)dt + \int_{-r}^{0} a(\sigma)A_2\int_0^\epsilon \widehat{u}(t+\sigma)dtd\sigma \\
&= \int_0^\epsilon\left(A\widehat{u}(t) + A_1\widehat{u}(t-r) + \int_{-r}^{0} a(\sigma)A_2\widehat{u}(t+\sigma)d\sigma\right)dt \\
&= \int_0^\epsilon \widehat{u}'(t)dt = \widehat{u}(\epsilon) - \widehat{u}(0) = \widehat{u}_\epsilon(0) - \varphi_1(0) = \varphi_\epsilon'(0).
\end{aligned}
$$

Therefore $\varphi_\epsilon \in D(\Lambda)$. Since $[0,\infty) \ni t \mapsto \widehat{u}_t \in Z$ is continuous,

$$
\|\epsilon^{-1}\varphi_\epsilon - \varphi_1\|_Z = \left\|\frac{1}{\epsilon}\int_0^\epsilon(\widehat{u}_t - \varphi_1)dt\right\|_Z \le \frac{1}{\epsilon}\int_0^\epsilon \|\widehat{u}_t - \varphi_1\|_Z dt \to 0 \text{ as } \epsilon \to 0.
$$

(iii) Let $\varphi_1 \in D(\Lambda)$ and u be the solution to (4.1). As was noted in the proof of (i) one has $\widehat{u} \in W^{1+\theta,p}(-r,\infty;D(A)) \cap C^1([-r,\infty];(X,D(A))_{\theta+1-1/p,p})$. Hence

$$
\widehat{u}' \in W^{\theta,p}(-r,\infty;D(A)) \cap C([-r,\infty];(X,D(A))_{\theta+1-1/p,p}).
$$

Therefore

$$\left\|\int_0^1 \widehat{u}'(t\sigma+\cdot)d\sigma - \widehat{u}'\right\|_{W^{\theta,p}(-r,0;D(A))} = \left\|\int_0^1 \left(\widehat{u}'(t\sigma+\cdot) - \widehat{u}'\right)d\sigma\right\|_{W^{\theta,p}(-r,0;D(A))}$$
$$\leq \int_0^1 \|\widehat{u}'(t\sigma+\cdot) - \widehat{u}'\|_{W^{\theta,p}(-r,0;D(A))}d\sigma \to 0,$$
$$\left\|\int_0^1 \widehat{u}'(t\sigma+\cdot)d\sigma - \widehat{u}'\right\|_{C([-r,0];(X,D(A))_{\theta+1-1/p,p})}$$
$$\leq \int_0^1 \|\widehat{u}'(t\sigma+\cdot) - \widehat{u}'\|_{C([-r,0];(X,D(A))_{\theta+1-1/p,p})}d\sigma \to 0.$$

Hence

$$\frac{S(t)\varphi_1-\varphi_1}{t} = \frac{\widehat{u}_t-\widehat{u}_0}{t} = \frac{\widehat{u}(t+\cdot)-\widehat{u}(\cdot)}{t} = \frac{1}{t}\int_0^1 \frac{\partial}{\partial\sigma}\widehat{u}(t\sigma+\cdot)d\sigma$$
$$= \int_0^1 \widehat{u}'(t\sigma+\cdot)d\sigma \to \widehat{u}'(\cdot) = \Lambda\varphi_1$$

in Z.
(iv) Let $\varphi_{1n} \in D(\Lambda),\ \varphi_{1n} \to \varphi_1,\ \varphi_{1n}' = \Lambda\varphi_{1n} \to \psi$ in Z. Then $\varphi_1' = \psi \in Z$. Hence

$$\varphi_1 \in W^{1+\theta,p}(-r,0;D(A)) \cap C^1([-r,0];(X,D(A))_{\theta+1-1/p,p}).$$

Since $\varphi_{1n} \to \varphi_1$ in $C([-r,0];(X,D(A))_{\theta+1-1/p,p})$, one has $A\varphi_{1n}(0) \to A\varphi_1(0)$, $A\varphi_{1n}(-r) \to A\varphi_1(-r)$, $\int_{-r}^0 a(\sigma)A_2\varphi_{1n}(\sigma)d\sigma \to \int_{-r}^0 a(\sigma)A_2\varphi_1(\sigma)d\sigma$ in $(X,D(A))_{\theta-1/p,p}$. One also has $\varphi_{1n}'(0) \to \psi(0) = \varphi_1'(0)$ in $(X,D(A))_{\theta+1-1/p,p}$. Therefore, from

$$\varphi_{1n}'(0) = A\varphi_{1n}(0) + A_1\varphi_{1n}(-r) + \int_{-r}^0 a(\sigma)A_2\varphi_{1n}(\sigma)d\sigma$$

it follows that

$$\varphi_1'(0) = A\varphi_1(0) + A_1\varphi_1(-r) + \int_{-r}^0 a(\sigma)A_2\varphi_1(\sigma)d\sigma.$$

Therefore $\varphi_1 \in D(\Lambda)$ and $\Lambda\varphi_1 = \psi$.

□

5. Remark on Hypothesis (II-2)

Let A be the realization in $L^p(\Omega), 1 < p < \infty$, of a strongly elliptic linear partial differential operator of second order with the Dirichlet boundary condition, where Ω is a bounded domain in $\mathbb{R}^n$. Let $A_i, i = 1,2$, be linear partial differential operators of second order in Ω. Assume that the coefficients of $A, A_i, i = 1,2$, and the boundary $\partial\Omega$ of Ω are sufficiently smooth. Then $D(A) = W^{2,p}(\Omega) \cap W_0^{1,p}(\Omega)$. Assume that A has a bounded inverse. Suppose $1/p < \theta < 3/(2p)$. Then $0 < 2\theta - 2/p < 1/p$. Hence by virtue of the results of R. Seeley [6] (also c.f. H. Triebel [7, Theorem 4.3.3]).

$$\begin{aligned}(L^p(\Omega), D(A))_{\theta-1/p,p} &= (L^p(\Omega), W^{2,p}(\Omega) \cap W_0^{1,p}(\Omega))_{\theta-1/p,p} = \\ &= B_{p,p}^{2\theta-2/p}(\Omega) = W^{2\theta-2/p,p}(\Omega).\end{aligned} \tag{5.1}$$

Since

$$A^{-1} \in \mathcal{L}(L^p(\Omega), W^{2,p}(\Omega) \cap W_0^{1,p}(\Omega)) \cap \mathcal{L}(W^{1,p}(\Omega), W^{3,p}(\Omega) \cap W_0^{1,p}(\Omega)),$$

one has

$$A^{-1} \in \mathcal{L}(W^{2\theta-2/p,p}(\Omega), W^{2+2\theta-2/p,p}(\Omega) \cap W_0^{1,p}(\Omega)). \tag{5.2}$$

Let $f \in (L^p(\Omega), D(A))_{\theta-1/p,p}$. In view of (5.1) and (5.2) $A^{-1}f \in W^{2+2\theta-2/p,p}(\Omega) \cap W_0^{1,p}(\Omega)$. Hence

$$A_i A^{-1} f \in W^{2\theta-2/p,p}(\Omega) = (L^p(\Omega), D(A))_{\theta-1/p,p}, \ i = 1, 2.$$

Therefore

$$A_i A^{-1} \in \mathcal{L}((L^p(\Omega), D(A))_{\theta-1/p,p}, (L^p(\Omega), D(A))_{\theta-1/p,p}), \ i = 1, 2. \tag{5.3}$$

Next, consider the case of the Neumann boundary condition. In this case

$$D(A) = \left\{ u \in W^{2,p}(\Omega); \partial u/\partial n = 0 \text{ on } \partial\Omega \right\},$$

where $\partial u/\partial n$ is the outer conormal derivative with respect to A. Suppose $1/p < \theta < 3/(2p) + 1/2$, $\theta \neq 1/2 + 1/p$. Then, again by virtue of the results of R. Seeley [6] or H. Triebel [7]

$$(L^p(\Omega), D(A))_{\theta-1/p,p} = B_{p,p}^{2(\theta-1/p)}(\Omega) = W^{2(\theta-1/p),p}(\Omega),$$

and (5.3) follows as in the case of the Dirichlet boundary condition.

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РЕЗУЛЬТАТЫ РЕГУЛЯРНОСТИ И РАЗРЕШАЮЩИХ ПОЛУГРУПП ДЛЯ ФУНКЦИОНАЛЬНО-ДИФФЕРЕНЦИАЛЬНЫХ УРАВНЕНИЙ С ЗАПАЗДЫВАНИЕМ

А. Фавини, Х. Танабе

Показано, что решения функционально-дифференциальных уравнений с запаздыванием в банаховом пространстве, существование и единственность которых показана раннее в работах А. Фавини и Х. Танабе, обладают дополнительными свойствами регулярности, если исходные данные и неоднородный член удовлетворяют некоторым предположениям о гладкости. Кроме того, получены некоторые результаты о разрешающих полугруппах.

Ключевые слова: функционально-дифференциальное уравнение с запаздыванием; регулярность решений; аналитической полугруппы; полугруппы решения; C_0-полугруппы; инфинитезимальный генератор.

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