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ANALYSIS OF THE WENTZELL STOCHASTIC SYSTEM COMPOSED OF THE EQUATIONS OF UNPRESSURISED FILTRATION IN THE HEMISPHERE AND AT ITS BOUNDARY

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The deterministic and stochastic Wentzell systems of Dzekzer equations in a hemisphere and on its boundary are studied for the first time. The deterministic case is characterised by the unambiguous solvability of the initial problem for the Wentzell system in a specific constructed Hilbert space. In the case of the stochastic hydrodynamic system "reservoir-well-collector", the theory of Nelson–Glicklich derivative is applied and a stochastic solution is constructed, which allows us to determine the prognoses of quantitative changes in the geochemical regime of groundwater under non-pressure filtration. It should be noted that for the filtration system under study, the non-classical Wentzell condition is considered, since it is represented by an equation with the Laplace – Beltrami operator defined on the boundary of the domain, understood as a smooth compact Riemannian manifold without an edge, and the external influence is represented by the normal derivative of the function defined in the domain.

Keywords: Wentzell system; Dzekzer equation; Nelson–Glicklich derivative.

Introduction

Let $\Omega \in \mathbb{R}^n$, $n \geq 2$, be a region with boundary Γ of the class C^∞. On a compact $\Omega \cup \Gamma$ we consider a system of two Dzekzer equations [1], modelling the evolution of the free surface of the filtering fluid

$$(\lambda - \Delta)u_t = \alpha_0 \Delta u - \beta_0 \Delta^2 u - \gamma_0 u, \ u = u(t,x), \ (t,x) \in \mathbb{R} \times \Omega, \tag{1}$$

$$(\lambda - \Delta)v_t = \alpha_1 \Delta v - \beta_1 \Delta^2 v + \frac{\partial u}{\partial \nu} - \gamma_1 v, \ v = v(t,x), \ (t,x) \in \mathbb{R} \times \Gamma, \tag{2}$$

$$\frac{\partial u}{\partial \nu} = 0, \ (t,x) \in \mathbb{R} \times \Gamma, \tag{3}$$

$$\mathrm{tr}\ u = v, \ \text{на}\ \mathbb{R} \times \Gamma. \tag{4}$$

The symbol Δ in (1) denotes the Laplace operator in the region Ω, and in (2) the same symbol denotes the Laplace – Beltrami operator on a smooth Riemannian manifold Γ. The symbol $\nu = \nu(t,x)$, $(t,x) \in \mathbb{R} \times \Gamma$ stands for the normal $\mathbb{R} \times \Gamma$ external to $\mathbb{R} \times \Omega$. The parameters $\alpha_0, \alpha_1, \lambda, \beta_0, \beta_1, \gamma_0, \gamma_1 \in \mathbb{R}$ describe the medium.

The condition of the form (2) and initial conditions (4) have been studied previously in various situations [2,3], so we will only give a brief history. It first appeared in [4] when constructing the Feller semigroup generator [5] for multidimensional diffusion processes in the bounded Ω region. In [6] it was shown for the first time that (2) arises naturally in biophysics to describe diffusion inside a cell and on its membrane. This approach to the study of problems where boundary conditions are treated not as limit values of the desired function and its derivatives, but as a description of some processes on the boundary, possibly only partially depending on the processes inside the region, led to the construction of a new direction in potential theory [7,8], where solutions of one-phase and two-phase Wentzell problems with the use of repeated double and simple layer potentials were obtained. Another approach is based on the ideas and methods of semigroup operator theory. In [9] it was first shown that the operator including the Laplace operator Δ inside the region Ω and the Laplace – Beltrami operator Δ on its boundary $\partial\Omega$ is a generator of a C_0-semigroup. In [10] this result was used in solving a number of applied problems. The first results of research in this direction were summarised in [11]. Moreover, in [12–15] analyticity conditions for solving C_0-continuous semigroups of operators were found. Finally, in [16] the case when the operator Δ s replaced by Δ^2 in Ω region, while on the boundary the Laplace – Beltrami operator Δ remains the same.

Our approach to the study of the problem (6) – (9) is unconventional – intending in the future to consider different cases of the domain Ω and the boundary Γ (for instance, Ω is a bounded connected Riemannian manifold with edge Γ) we consider it necessary to call (1), (2) a system of equations, albeit defined on sets of different geometric dimension. This approach is supported by the fact that equations (1) and (2) describe the same physical process of moisture filtration. The term "boundary conditions" should be reserved for equations defined on the boundary (edge) of a region (manifold) and having a lower order of derivatives on spatial variables (see the classical treatise [17]).

In the simplest case, we will study the solvability of system (1) – (3): $\Omega = \{(r,\theta,\varphi) : r \in [0,R), \theta \in [0,\frac{\pi}{2}], \varphi \in [0,2\pi]\}$ in $\mathbb{R}^3$, but $\Gamma = \{(\theta,\varphi) : \theta \in [0,\frac{\pi}{2}), \varphi \in [0,2\pi)\}$ is a hemisphere with boundary. In this case, (1) – (3) is transformed to the form

$$(\lambda - \Delta_{r,\theta,\varphi})u_t = \alpha_0\Delta_{r,\theta,\varphi}u - \beta_0\Delta^2_{r,\theta,\varphi}u - \gamma_0 u,\ u = u(t,r,\theta,\varphi),\ (t,r,\theta,\varphi) \in \mathbb{R}\times\Omega, \quad (5)$$

$$(\lambda - \Delta_{\theta,\varphi})v_t = \alpha_1\Delta_{\theta,\varphi}v - \beta_1\Delta^2_{r,\theta,\varphi}v + \partial_R u - \gamma_1 v,\ v = v(t,\theta,\varphi),\ (t,\theta,\varphi) \in \mathbb{R}\times\Gamma, \quad (6)$$

$$\partial_R u = 0,\ v = v(t,\theta,\varphi),\ (t,\theta,\varphi) \in \mathbb{R}\times\Gamma, \quad (7)$$

where

$$\begin{aligned}
\Delta_{r,\theta,\varphi} &= (r-R)\frac{\partial}{\partial r}\left((R-r)\frac{\partial}{\partial r}\right) + \frac{\partial^2}{\partial\theta^2} + \frac{\partial^2}{\partial\varphi^2},\\
\Delta_{\theta,\varphi} &= \frac{\partial^2}{\partial\theta^2} + \frac{\partial^2}{\partial\varphi^2}, \quad \partial_R = \frac{\partial}{\partial r}\bigg|_{r=R},\\
\Delta_{\theta,\varphi} &= \frac{\partial^2}{\partial\theta^2} + \frac{\partial^2}{\partial\varphi^2}, \quad \partial_R = \frac{\partial}{\partial r}\bigg|_{r=R}.
\end{aligned} \quad (8)$$

To the given system we add the matching condition (4) and equip it with initial conditions

$$u(0,r,\theta,\varphi) = u_0(r,\theta,\varphi), \quad v(0,\theta,\varphi) = v_0(\theta,\varphi). \quad (9)$$

Let us call the solution of the problem (4) – (9) a deterministic solution of the Wentzell system. We note that by transforming the operator (17) to Cartesian coordinates we obtain

$$\Delta_{x,y} = x\frac{\partial}{\partial x} + y\frac{\partial}{\partial y} + z\frac{\partial}{\partial z} + \left(z^2 + y^2\right)\frac{\partial^2}{\partial x^2} + \left(\frac{xy(x^2+y^2+z^2)}{x^2+y^2} - 2xy\right)\frac{\partial^2}{\partial x\partial y} + \\ + \left(x^2 + y^2\frac{x^2+y^2+z^2}{x^2+y^2}\right)\frac{\partial^2}{\partial y^2} + \left(x^2+y^2+z^2\right)\frac{\partial^2}{\partial z^2}.$$

We shall transfer the consideration of the Laplace operator in standard spherical coordinates to our future studies.

In addition to the introduction and the list of references, the paper contains two parts. The first part considers the existence and uniqueness of a deterministic system of Wentzell equations in a hemisphere and on its boundary. The second part contains abstract reasoning consisting in the construction of the space of ($\mathfrak{H}$-*valued*) $\mathbf{K}$-*"noises"* and the proof of existence and uniqueness of the stochastic system of Wentzell equations in the hemisphere and on its boundary.

1. Deterministic Wentzell System

Let us consider the following series

$$u = \sum_{k=2}^{\infty}\exp\left(t\frac{-\beta_0k^4-\alpha_0k^2-\gamma_0}{\lambda+k^2}\right)\frac{(R-r)^k}{R^k}\Big(a_k\sin k\theta(\sin k\varphi+\cos k\varphi)+b_k\cos k\theta(\sin k\varphi+ \\ +\cos k\varphi)\Big) + \sum_{k=1}^{\infty}\exp\left(t\frac{-\beta_0k^4-\alpha_0k^2-\gamma_0}{\lambda+k^2}\right)\Big(c_k\sin k\theta(\sin k\varphi+\cos k\varphi)+ \\ +d_k\cos k\theta(\sin k\varphi+\cos k\varphi)\Big), \tag{10}$$

where

$$a_k = \int_0^{2\pi}d\varphi\int_0^R u_0(r,\theta,\varphi)\frac{(R-r)^k}{R^k}\sin k\theta(\sin k\varphi+\cos k\varphi)rdr,$$

$$b_k = \int_0^{2\pi}d\varphi\int_0^R u_0(r,\theta,\varphi)\frac{(R-r)^k}{R^k}\cos k\theta(\sin k\varphi+\cos k\varphi)rdr,$$

$$c_k = \int_0^{2\pi}d\varphi\int_0^{\frac{\pi}{2}} v_0(\theta,\varphi)\sin k\theta(\sin k\varphi+\cos k\varphi)d\theta,$$

$$d_k = \int_0^{2\pi}d\varphi\int_0^{\frac{\pi}{2}} v_0(\theta,\varphi)\cos k\theta(\sin k\varphi+\cos k\varphi)d\theta.$$

It is not complicated to notice that the constructed series above is a formal solution of (5). Furthermore, if the series in (10) converges uniformly, then we have a solution of the problem (5), (9), where $\partial_R u = 0$, $\Delta u = 0$. Taking this into account, we can construct a solution of problem (6), (9)

$$v = \sum_{k=1}^{\infty}\exp\left(t\frac{-\beta_1k^4-\alpha_1k^2-\gamma_1}{\lambda+k^2}\right)\Big(c_{k,n}\cos k\varphi+d_{k,n}\sin k\varphi\Big), \tag{11}$$

where in the case $\alpha_0 = \alpha_1$, $\beta_0 = \beta_1$, $\gamma_0 = \gamma_1$ the solutions of the problem (6) – (9) will satisfy the matching condition (4).

The closure of the lineal span$\{(R^k)^{-1}(R-r)^k \sin k\theta(\sin k\varphi + \cos k\varphi),\ (R^k)^{-1}(R-r)^k \cos k\theta\cdot\ \cdot(\sin k\varphi + \cos k\varphi)\colon k \in \mathbb{N}\backslash\{1\}, r \in (0,R), \theta \in [0,\frac{\pi}{2}], \varphi \in [0,2\pi)\}$ generated by the scalar product

$$(\varphi,\psi) = \int\limits_0^R dr \int\limits_0^{2\pi} d\varphi \int\limits_0^{\frac{\pi}{2}} \varphi(r,\theta,\varphi)\psi(r,\theta,\varphi) r^2 \sin\theta d\theta,$$

we denote by the symbol $A(\Omega)$. Then, the closure of the lineal span$\{\sin k\theta(\sin k\varphi + \cos k\varphi),\ \cos k\theta(\sin k\varphi + \cos k\varphi)\colon k \in \mathbb{N}, \theta \in [0,\frac{\pi}{2}], \varphi \in [0,2\pi)\}$ by the norm generated by the scalar product

$$(\varphi,\psi) = \int\limits_0^{2\pi} d\varphi \int\limits_0^{\frac{\pi}{2}} \varphi(r,\theta,\varphi)\psi(r,\theta,\varphi) d\theta,$$

we denote with the symbol $A(\Gamma)$.

Thus, the following theorem occurs.

Theorem 1. *For any* $u_0 \in A(\Omega)$ *and* $v_0 \in A(\Gamma)$ *such that* (4) *is satisfied, and for any coefficients* $\alpha_0, \alpha_1, \lambda, \beta_0, \beta_1, \gamma_0, \gamma_1 \in \mathbb{R}$*, such that the following condition is satisfied* $\alpha_0 = \alpha_1$*,* $\beta_0 = \beta_1$*,* $\gamma_0 = \gamma_1$*, a* $\lambda \neq k^2$*, where* $k \in \mathbb{N}$*, cthere exists a single solution* $(u,v) \in C^\infty A(\Omega) \oplus A(\Gamma))$ *of the problem* (4) – (9).

2. Stochastic Wentzell System

Let $\Omega \equiv (\Omega, \mathcal{A}, \mathbf{P})$ be a complete probability space with probability measure $\mathbf{P}$, associated with the σ-algebra $\mathcal{A}$ of subsets of the set Ω, and let $\mathbb{R}$ be the set of real numbers endowed with a Borel σ-algebra. A measurable mapping $\xi : \Omega \to \mathbb{R}$ is called *a random variable*. The set of random variables with zero expectation and finite variance forms a Hilbert space $\mathbf{L}_2$ with scalar product $(\xi_1, \xi_2) = \mathbf{E}\xi_1\xi_2$.

Let $\mathfrak{I} \subset \mathbb{R}$ be an internal. We call the measurable mapping $\eta : \mathfrak{I} \times \Omega \to \mathbb{R}$, *a stochastic process*, for each fixed $\omega \in \Omega$ the function $\eta(\cdot,\omega) : \mathfrak{I} \to \mathbb{R}$ is *its trajectory*, and for each fixed $t \in \mathfrak{I}$ the random variable $\eta(t,\cdot) : \Omega \to \mathbb{R}$ is *its cross section*. We call a stochastic process $\eta = \eta(t)$, $t \in \mathfrak{I}$, *continuous stochastic process* if almost probably all its trajectories are continuous (i.e. if almost all $\omega \in \mathcal{A}$ the trajectories $\eta(\cdot,\omega)$ are continuous functions). A multitude of continuous stochastic processes forms a Banach space, which we denote by the symbol $\mathbf{C}(\mathfrak{I}; \mathbf{L}_2)$ with norm

$$\|\eta\|_{\mathbf{C}L_2} = \sup_{t \in \mathfrak{I}} (\mathbf{D}\eta(t,\omega))^{1/2}.$$

Let $\mathcal{A}_0$ be an σ-subalgebra of σ-algebra $\mathcal{A}$. Let us construct a subspace $\mathbf{L}_2^0 \subset \mathbf{L}_2$ of random variables. of random variables measurable with respect to $\mathcal{A}_0$. We denote by $\Pi : \mathbf{L}_2 \to \mathbf{L}_2^0$ the orthoprojector. Let $\xi \in \mathbf{L}_2$, then $\Pi\xi$ is called *the conditional mathematical expectation* of the random variable ξ is denoted by $\mathbf{E}(\xi|\mathcal{A}_0)$. We fix $\eta \in \mathbf{C}(\mathfrak{I}; \mathbf{L}_2)$ and $t \in \mathfrak{I}$, denote by $\mathcal{N}_t^\eta$ *the* σ*-algebra, generated by the random variable* $\eta(t)$, and define $\mathbf{E}_t^\eta = \mathbf{E}(\cdot|\mathcal{N}_t^\eta)$.

Definition 1. *Let $\eta \in \mathbf{C}(\mathfrak{I};\mathbf{L}_2)$. The derivative of the Nelson–Glicklich process $\overset{\circ}{\eta}$ of stochastic process η at a point $t \in \mathfrak{I}$ is a random variable*

$$\overset{\circ}{\eta}(t,\cdot) = \frac{1}{2}\left(\lim_{\Delta t \to 0+} \mathbf{E}_t^{\eta}\left(\frac{\eta(t+\Delta t,\cdot)-\eta(t,\cdot)}{\Delta t}\right) + \right.$$

$$\left. + \lim_{\Delta t \to 0+} \mathbf{E}_t^{\eta}\left(\frac{\eta(t,\cdot)-\eta(t-\Delta t,\cdot)}{\Delta t}\right)\right),$$

if the limit exists in the sense of a uniform metric on $\mathbb{R}$.

If the Nelson – Glickich derivatives $\overset{\circ}{\eta}(t,\cdot)$ of the stochastic process $\eta(t,\cdot)$ exist at all (or p.c.) points of the interval $\mathfrak{I}$, then we say that the Nelson–Glicklich derivative $\overset{\circ}{\eta}(t,\cdot)$ on $\mathfrak{I}$ (p.c. on $\mathfrak{I}$). The set of continuous stochastic processes having continuous Nelson–Glicklich derivative $\overset{\circ}{\eta}$ orm a Banach function $\mathbf{C}^1(\mathfrak{I};\mathbf{L}_2)$ space with norm

$$\|\eta\|_{\mathbf{C}^1\mathbf{L}_2} = \sup_{t\in\mathfrak{I}}\left(\mathbf{D}\eta(t,\omega) + \mathbf{D}\overset{\circ}{\eta}(t,\omega)\right)^{1/2}.$$

Let us further define by induction the Banach spaces $\mathbf{C}^l(\mathfrak{I};\mathbf{L}_2)$, $l \in \mathbb{N}$, of stochastic processes whose trajectories are Nelson–Glicklich differentiable on $\mathfrak{I}$ up to order $l \in \{0\}\cup\mathbb{N}$ inclusive. Their norms are given by the formulas

$$\|\eta\|_{\mathbf{C}^l\mathbf{L}_2} = \sup_{t\in\mathfrak{I}}\left(\sum_{k=0}^{l}\mathbf{D}\overset{\circ}{\eta}{}^{(k)}(t,\omega)\right)^{1/2}.$$

Here we will consider the zero-order Nelson–Glicklich derivative as the initial random process, e.g. $\overset{\circ}{\eta}{}^{(0)} \equiv \eta$. Note also that the spaces $\mathbf{C}^l(\mathfrak{I};\mathbf{L}_2)$, $l \in \{0\}\cup\mathbb{N}$, for the sake of for brevity we will call *the spaces of "noise"*.

Let us proceed to the construction of the space of *random* $\mathbf{K}$*-values.* Let $\mathfrak{H}$ be a real separable Hilbert space with orthonormalised basis $\{\varphi_k\}$, monotone sequence $\mathbf{K} = \{\lambda_k\} \subset \mathbb{R}_+$, such that $\sum\limits_{k=1}^{\infty}\lambda_k^2 < +\infty$, and also the sequence $\{\xi_k\} = \xi_k(\omega) \subset \mathbf{L}_2$ random variables such that $\|\xi_k\|_{\mathbf{L}_2} \leq C$, for some constant $C \in \mathbb{R}_+$ and for all $k \in \mathbb{N}$. Let us construct an $\mathfrak{H}$*-valued random* $\mathbf{K}$*-values*

$$\xi(\omega) = \sum_{k=1}^{\infty}\lambda_k\xi_k(\omega)\varphi_k.$$

Completion of the linear envelope of the set $\{\lambda_k\xi_k\varphi_k\}$ by norm

$$\|\eta\|_{\mathbf{H}_K\mathbf{L}_2} = \left(\sum_{k=1}^{\infty}\lambda_k^2\mathbf{D}\xi_k\right)^{1/2}$$

is called *the space of ($\mathfrak{H}$-valued) random* $\mathbf{K}$*-values* and is denoted by the symbol $\mathbf{H}_K\mathbf{L}_2$. It should be obvious that the space $\mathbf{H}_K\mathbf{L}_2$ is Hilbertian, with the random $\mathbf{K}$-value $\xi = \xi(\omega) \in \mathbf{H}_K\mathbf{L}_2$. Equivalently, we define the Banach space of ($\mathfrak{H}$*-valued*) $\mathbf{K}$-*"noises"* $\mathbf{C}^l(\mathfrak{I};\mathbf{H}_K\mathbf{L}_2)$, $l \in \{0\}\cup\mathbb{N}$, to be an enlargement of the linear envelope of the set $\{\lambda_k\eta_k\varphi_k\}$ by norm

$$\|\eta\|_{\mathbf{C}^l\mathbf{H}_K\mathbf{L}_2} = \sup_{t\in\mathfrak{I}}\left(\sum_{k=1}^{\infty}\lambda_k^2\sum_{m=1}^{l}\mathbf{D}\overset{\circ}{\eta}{}_k^{(m)}\right)^{1/2},$$

where the sequence of "noises" $\{\eta_k\} \subset \mathbf{C}^l(\mathfrak{I}; \mathbf{L}_2)$, $l \in \{0\} \cup \mathbb{N}$. Obviously, the vector

$$\eta(t,\omega) = \sum_{k=1}^{\infty} \lambda_k \eta_k(t,\omega)\varphi_k$$

lies in the space $\mathbf{C}^l(\mathfrak{I}; \mathbf{H}_K\mathbf{L}_2)$, if the sequence of vectors $\{\eta_k\} \subset \mathbf{C}^l(\mathfrak{I}; \mathbf{L}_2)$ and all their Nelson–Glicklich derivatives up to and including order $l \in \{0\} \cup \mathbb{N}$ are uniformly bounded in norm $\|\cdot\|_{\mathbf{C}^l\mathbf{L}_2}$.

Example. Vector lying in all spaces $\mathbf{C}^l(\mathbb{R}_+; \mathbf{H}_K\mathbf{L}_2)$, $l \in \{0\} \cup \mathbb{N}$,

$$W_{\mathbf{K}}(t,\omega) = \sum_{k=1}^{\infty} \lambda_k \beta_k(t,\omega)\varphi_k,$$

where $\{\beta_k\} \subset \mathbf{C}^l(\mathfrak{I}; \mathbf{L}_2)$ is a sequence of Brownian motions, is called an *($\mathfrak{H}$-valued) Wiener* $\mathbf{K}$*-process.*

Let $\mathfrak{U}$ ($\mathfrak{F}$) now be a real separable Hilbert space with orthonormalised basis $\{\varphi_k\}$ ($\{\psi_k\}$). Let us introduce a monotone sequence $\mathbf{K} = \{\lambda_k\} \subset \{0\} \cup \mathbb{R}$ such that $\sum\limits_{k=1}^{\infty} \lambda_k^2 < +\infty$. By the symbol $\mathbf{U_K L}_2$ ($\mathbf{F_K L}_2$) we denote the Hilbert space. which is a replenishment of the linear envelope of *random* $\mathbf{K}$*-values*

$$\xi = \sum_{k=1}^{\infty} \lambda_k \xi_k \varphi_k,\ \xi_k \in \mathbf{L}_2, \quad \left(\zeta = \sum_{k=1}^{\infty} \mu_k \zeta_k \psi_k,\ \zeta_k \in \mathbf{L}_2\right),$$

by norm

$$\|\eta\|^2_{\mathbf{U}} = \sum_{k=1}^{\infty} \lambda_k^2 \mathbf{D}\xi_k \quad \left(\|\omega\|^2_{\mathbf{F}} = \sum_{k=1}^{\infty} \mu_k^2 \mathbf{D}\zeta_k\right).$$

Note that in different spaces ($\mathbf{U_K L}_2$ и $\mathbf{F_K L}_2$) the sequence $\mathbf{K}$ can be different ($\mathbf{K} = \{\lambda_k\}$ and $\mathbf{U_K L}_2$ и $\mathbf{K} = \{\mu_k\}$ в $\mathbf{F_K L}_2$), but all sequences marked by $\mathbf{K}$, must be monotone and summable with square. All results will generally be true for different sequences $\{\lambda_k\}$ and $\{\mu_k\}$, but for the sake of simplicity we will restrict ourselves to the case $\lambda_k = \mu_k$.

Let $A : \mathfrak{U} \to \mathfrak{F}$ be a linear operator. By the formula

$$A\xi = \sum_{k=1}^{\infty} \lambda_k \xi_k A\varphi_k \tag{12}$$

we define a linear operator $A : \mathbf{U_K L}_2 \to \mathbf{F_K L}_2$, and if the series in the right-hand side of (12) converges (in the $\mathbf{F_K L}_2$ metric), then $\xi \in \mathrm{dom}A$, and if diverges, then $\xi \notin \mathrm{dom}A$.Traditionally the spaces of linear continuous operators $\mathcal{L}(\mathbf{U_K L}_2; \mathbf{F_K L}_2)$ and linear closed densely defined operators are traditionally defined. The following holds

Lemma 1. *(i) Operator $A \in \mathcal{L}(\mathfrak{U}; \mathfrak{F})$ is exactly and only if $A \in \mathcal{L}(\mathbf{U_K L}_2; \mathbf{F_K L}_2)$.*

Since it is clear to see,

$$\|A\xi\|_{\mathbf{F}} \le \sum_{k=1}^{\infty} \lambda_k^2 \mathbf{D}\xi_k \|A\varphi_k\|^2_{\mathfrak{F}} \le \mathrm{const} \sum_{k=1}^{\infty} \lambda_k^2 \mathbf{D}\xi_k = \mathrm{const}\, \|\xi\|_{\mathbf{U}}.$$

(ii) Operator $A \in Cl(\mathfrak{U};\mathfrak{F})$ is exactly and only if $A \in Cl(\mathbf{U_K L}_2;\mathbf{F_K L}_2)$.

For reasons of simplicity, let $\mathfrak{U} = \{u \in W_2^2(\Omega) \oplus W_2^2(\Gamma) : \partial_R u = 0\}$, $\mathfrak{F} = L_2(\Omega) \oplus L_2(\Gamma)$. Following the algorithm outlined above, we then construct the spaces of *random* **K**-*values*. A *random* **K**-*value* $\xi \in \mathbf{U_K L}_2$ has the following form

$$\xi = \sum_{k=1}^{\infty} \lambda_k \xi_k \varphi_k, \tag{13}$$

where $\{\varphi_k\}$ is the family of eigenfunctions of the modified Laplace operator $\Delta_{r,\theta,\varphi} \in \mathcal{L}(\mathfrak{U};\mathfrak{F})$ orthonormalised in the sense of the scalar product $(\cdot,\cdot)$ from $L_2(\Omega)$. Let us consider the linear stochastic Wentzell system of the moisture filtration equation in the balloon and at its boundary. In this case (1), (2) is transformed to the form

$$(\lambda - \Delta_{r,\theta,\varphi})\eta_t = \alpha_0 \Delta_{r,\theta,\varphi}\eta - \beta_0 \Delta^2_{r,\theta,\varphi}\eta - \gamma_0\eta, \eta \in C^\infty(\mathbb{R}_+;\mathbf{U_K L}_2), \tag{14}$$

$$(\lambda - \Delta_{\theta,\varphi})\eta_t = \alpha_1 \Delta_{\theta,\varphi}\eta - \beta_1 \Delta^2_{r,\theta,\varphi}\eta + \partial_R\eta - \gamma_1\eta, \eta \in C^\infty(\mathbb{R}_+;\mathbf{U_K L}_2), \tag{15}$$

$$\partial_R \eta = 0, \eta \in C^\infty(\mathbb{R}_+;\mathbf{U_K L}_2), \tag{16}$$

where

$$\begin{aligned} \Delta_{r,\theta,\varphi} &= (r-R)\frac{\partial}{\partial r}\left((R-r)\frac{\partial}{\partial r}\right) + \frac{\partial^2}{\partial\theta^2} + \frac{\partial^2}{\partial\varphi^2}, \\ \Delta_{\theta,\varphi} &= \frac{\partial^2}{\partial\theta^2} + \frac{\partial^2}{\partial\varphi^2}, \quad \partial_R = \left.\frac{\partial}{\partial r}\right|_{r=R}, \\ \Delta_{\theta,\varphi} &= \frac{\partial^2}{\partial\theta^2} + \frac{\partial^2}{\partial\varphi^2}, \quad \partial_R = \left.\frac{\partial}{\partial r}\right|_{r=R}. \end{aligned} \tag{17}$$

For this system we add a matching condition and equip it with initial conditions

$$\eta(0) = \eta_0 \tag{18}$$

The solution of the problem (14) – (18) we call the stochastic solution of the Wentzell system.

Theorem 2. *For any $\eta_0 \in \mathbf{U_K L}_2(\Omega)$ and for coefficients $\alpha_0, \alpha_1, \lambda, \beta_0, \beta_1, \gamma_0, \gamma_1 \in \mathbb{R}$, such that the following condition $\alpha_0 = \alpha_1$, $\beta_0 = \beta_1$, $\gamma_0 = \gamma_1$, and $\lambda \neq k^2$ is satisfied, where $k \in \mathbb{N}$, there exists a single solution $\eta \in C^\infty(\mathbb{R}_+;\mathbf{U_K L}_2)$ of the stochastic Wentzell system (14) – (18).*

Proof. The existence and singularity of the solution are proved by analogy with the deterministic case due to the validity of Lemma 1.

□

Conclusion

We constructed the resolution group in the Cauchy–Wentzell system in the hemisphere and its biundary. Further, we plan to continue the results of the paper by applying the Wentzell conditions in directions related to [18–20].

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АНАЛИЗ СТОХАСТИЧЕСКОЙ СИСТЕМЫ ВЕНТЦЕЛЯ, СОСТАВЛЕННОЙ ИЗ УРАВНЕНИЙ БЕЗНАПОРНОЙ ФИЛЬТРАЦИИ В ПОЛУСФЕРЕ И НА ЕГО ГРАНИЦЕ

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Впервые изучены детерминированная и стохастическая системы Вентцеля уравнений Дзекцера в полусфере и на его границе. В детерминированном случае установлена однозначная разрешимость начальной задачи для системы Вентцеля в специфическом построенном гильбертовом пространстве. В случае стохастической гидродинамической системы «пласт – скважина – коллектор» используется теория производной Нельсона – Гликлиха и строится стохастическое решение, которое позволяет определять прогнозы количественного изменения геохимического режима грунтовых вод при безнапорной фильтрации. Отметим, что для изучаемой системы фильтрации рассматривалось неклассическое условие Вентцеля, поскольку оно представлено уравнением с оператором Лапласа – Бельтрами, заданным на границе области, понимаемой как гладкое компактное риманово многообразие без края, причем внешнее воздействие представлено нормальной производной функции, заданной в области.

Ключевые слова: система Вентцеля; уравнение Дзекцера; производная Нельсона – Гликлиха.

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