

## ON THE MEASUREMENT OF THE «WHITE NOISE»

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In the framework of the Leontieff type equations theory we consider the mathematical model of the measuring transducer, demonstrating the mechanical lag effect. In studying the model with deterministic external signal the methods and results of the Sobolev type equations theory and degenerate groups of operators are very useful, because they help to create an efficient computational algorithm. Now, the model assumes the presence of white noise along with the deterministic signal. Since the model is represented by a degenerate system of ordinary differential equations, it is difficult to apply existing nowadays approaches such as Ito – Stratonovich – Skorohod and Melnikova – Filinkov – Alshansky in which the white noise is understood as a generalized derivative of the Wiener process. Instead, we propose a new concept of the «white noise», which is equal to the symmetric mean derivative (in the paper – the derivative of the Nelson – Gliklikh) of the Wiener process, and in the framework of the Einstein – Smoluchowsky coincides with the «ordinary» derivative of the Brownian motion. The first part of the paper contains the basic facts of the Nelson – Gliklikh derivative theory adapted to this situation. The second part deals with the weakened Showalter – Sidorov problem and gives exact formulas for its solution. As an example, we present a concrete model of a measuring transducer.

*Keywords: Leontieff type equations, weakened Showalter – Sidorov problem, symmetric mean derivative, Wiener process.*

## Introduction

One of the authors [1] proposes a mathematical model of the measuring transducer (MT), and the co-author [2] proposed to consider it as a part of Leontieff type equations

$$L\dot{x} = Mx + y, \quad (1)$$

titled for obvious similarity with the balance Leontieff equations. Model (1) is proved to be adequate to a wide range of measured phenomena [3], [4]; obtained from its data agreed well with the results of field experiments. Later, on the basis of the model (1) the authors develop a method of signal restoration, distorted by mechanical lag [5], and the resonance in the circuits of MT [6]. On the basis of this method a numerical algorithm for reconstructing a signal distorted by lag effect of MT [7] has been constructed.

In (1)  $L$ ,  $M$  are square matrixes of order  $n$ , and  $\det L = 0$ . If we use a well-known theory of Kronecker – Weierstrass (see for example [8], Chapter 12), then in the case of a regular pencil  $L + \lambda M$  system (1) can be reduced to the equivalent system

$$\tilde{L}\dot{\tilde{x}} = \tilde{M}\tilde{x} + \tilde{y}, \quad (2)$$

where matrices  $\tilde{L} = \text{diag}\{N_{\nu_1}, N_{\nu_2}, \dots, N_{\nu_k}, \mathbb{I}_l\}$ ,  $\tilde{M} = \text{diag}\{\mathbb{I}_m, S\}$ ,  $N_{\nu_j}$  are Jordan boxes of order  $\nu_j$ ,  $j = \overline{1, k}$ , with zeros on main diagonals;  $\mathbb{I}_l$  and  $\mathbb{I}_m$  are identity matrices,  $l = n - m$ ,  $m = \sum_{j=1}^k \nu_j$ ;  $S$  is a square matrix of order  $l$ . In (2)  $m$  components of the vector function  $\tilde{x} = \tilde{x}(t)$  correspond to

the output signal and the remaining components characterize the condition the MT: the vector-function  $\tilde{y} = \tilde{y}(t)$  models the input signal. The same must be said about their pre-images from (1). Algorithm of reducing of (1) to (2) in the theory is very simple, but very unstable in the numerical implementation. Therefore, to investigate (1) we use the results of the degenerate operator groups theory [9], Chapter 4. Such approach lets us to create not only a numerical algorithm for solving the problems with inertia and resonances of MT, but also to construct complexes of programs [10, 11].

In all our previous works it has been tacitly assumed that the input signal  $y = y(t)$  is determined, that is, it does not contain random disturbances, such as additive white noise. So far as it is difficult to imagine the MT without the white noise (WN), we intend to study this phenomenon in the model (1), where the input signal is not only the useful signal, but it's also the WN. However, before we start this project, we want to make few remarks about the WN as a phenomenon.

It is believed that the history of the study of white-noise is based on the theory of the Brownian motion of A. Einstein and M. Smoluchowsky. From this theory it follows that the displacement of a particle in the Brownian motion is proportional to the  $\sqrt{t}$ , where  $t$  is time. Therefore, the particle velocity is proportional to the  $(2\sqrt{t})^{-1}$  and therefore it is not defined at the instant time  $t = 0$ . The next step in this direction is made by N. Wiener, who has suggested that the displacement of a particle is determined by a random process, which later got his name. Thus, the *Wiener process* is the random process  $w(t)$ , and has the following properties:

- (w1)  $w(0) = 0$  almost surely (a.s.), and sample paths  $w(t)$  are a.s. continuous;
- (w2) mathematical expectation  $E(w(t)) = 0$ , and autocorrelation function  $E((w(t) - w(s))^2) = |t - s|$ ;
- (w3) sample paths  $w(t)$  are a.s. nondifferentiable for all  $t \in [0, +\infty)$  and on any arbitrarily small interval have unbounded variation.

Usually, under the *white noise* we understand the generalized derivative of the Wiener process (since «regular» derivative does not exist due to (w3)). In this sense the white noise is presented, for example, in a linear stochastic differential equation

$$dx = (Sx + y)dt + A\delta w. \tag{3}$$

Here, in the right side, the symbol  $\delta w$  means the generalized differential of the Wiener process  $w(t)$ , that is the WN. The equations of the form (3) are studied for the first time by K. Ito [12], and then R.L. Stratonovich [13] and A.V. Skorokhod [14] joined the research. Their approaches differ mainly in the interpretation of the integral  $\int_0^\tau A\delta w(t)$ , which appears in the right-hand side of (3) after the integration. At the present time this approach is extended to partial differential equations [15]. In addition, recently in I.V. Melnikova school [16] a new direction of stochastic differential equations theory is formed and is actively developing. Here, the equation (3) is understood in the form

$$\dot{x} = Sx + y + \dot{w}, \tag{4}$$

where all derivatives are considered in the Schwartz space. Thus, the generalized derivative of the Wiener process in the right-hand side of (4) is an additive WN.

However, the approaches of Ito – Stratonovich – Skorokhod and Melnikova – Filinkov – Alshansky can hardly be applied to the equations of the form (1) research. In the first approach as it follows from the Kronecker – Weierstrass theory equations (1) (or in the equivalent form (2)) are divided into two parts; one part can be solved by the integration, such as (3), but the other part can be solved only by the repeated differentiation. And if the first derivative of the Wiener process has caused so much debate, what can we say about the second, the third and etc.?! The second approach is inapplicable due to the fact that the authors develop a theory

of optimal dynamic measurements based on the theory of the optimal control of equations (1) solutions, but it exists now only in the framework of Hilbert spaces, and nothing is heard about its distribution on a locally convex space (as in (4)).

The solution is seen by authors in the usage of the mean derivative instead of the generalized derivative of the Wiener process. E. Nelson [17] laid the fundamentals of the derivatives theory, and then Yu.E. Gliklikh [18] developed the theory till to its current state. One of the most important objects of this theory is the symmetric mean derivative of the random process, which is also called as the current rate of this process. Further the short, this very derivative will be called as the *derivative of Nelson – Gliklikh*, but we will keep the author's notations. For example, the Nelson – Gliklikh derivative of the Wiener process  $w(t)$  we will designate by symbol  $D_S w(t)$ .

Let's enumerate the advantages of such substitution. Firstly, in the case of deterministic (i.e. non-random) smooth process, the Nelson – Gliklikh derivative coincides with the «ordinary» derivative just as the generalized derivative coincides with the «ordinary» derivative of the smooth function. Secondly, the Nelson – Gliklikh derivative of the Wiener process  $w(t)$  has calculated and has been the following form:  $D_S w(t) = (2t)^{-1}w(t)$ . This random process we call the «white noise» («WN»), paying attention to the quotes. As well as the generalized derivative of the Wiener process, our «WN» due to (w2) has zero expectation. Finally, if the Wiener process  $w(t)$  simulates the displacement of a particle in the Brownian motion, then, according to Einstein – Smoluchowsky theory, its sample paths are a. s. equivalent to  $\sqrt{t}$ . From this  $D_S w(t)$  is a. s. equivalent to  $(2\sqrt{t})^{-1}$ , which coincides with the «ordinary» derivative of Brownian motion.

This paper is organized as follows. The first part contains the basic facts of Nelson – Gliklikh derivative theory adapted to our situation. Here we follow mainly [18]. In the second part we investigate the Showalter – Sidorov problem for the equations of the form (1) with the additive «WN» in the right side. Moreover, this problem is taken as a «weakened» in the sense of S.G. Krein. We show the existence of this problem solutions and give the exact formula. As an example, the specific model of the WN is given. As we repeatedly note the benefits of the Showalter – Sidorov problem compared to the Cauchy problem for Leontieff type equations (and even more general Sobolev type), refer the interested reader to [7, 19].

Finally, the authors consider their wish to express their gratitude to Yu.E. Gliklikh for his strict but constructive criticism.

## 1. The Nelson – Gliklikh derivative

Let  $\Omega = (\Omega, \mathcal{A}, \mathbf{P})$  be complete probability space,  $\mathbb{R}^n$  – dimensional vector space endowed with the Borel  $\sigma$ -algebra. Let's call the mapping  $\xi : \Omega \rightarrow \mathbb{R}^n$  as the *random value*, and denote the set of random variables by the symbol  $\mathcal{V}(\Omega; \mathbb{R}^n)$ . In this set we select a Lebesgue space  $L_q(\Omega; \mathbb{R}^n)$ ,  $q \in [0, +\infty)$ , and note that the inclusion  $L_q(\Omega; \mathbb{R}^n) \hookrightarrow L_r(\Omega; \mathbb{R}^n)$  is dense and continuous, if  $q \geq r$ , and the set  $\Omega$  is bounded. Suppose further that  $\mathcal{A}_0$  is the certain  $\sigma$ -algebra on  $\Omega$ , and  $\mathcal{A}_0 \subset \mathcal{A}$ ; by  $\Omega_0 = (\Omega, \mathcal{A}_0, \mathbf{P})$  denote corresponding complete probability space. It is easily seen that,  $L_2(\Omega_0; \mathbb{R}^n)$  is closed space in the  $L_2(\Omega; \mathbb{R}^n)$ . By  $\Pi : L_2(\Omega; \mathbb{R}^n) \rightarrow L_2(\Omega_0; \mathbb{R}^n)$  denote the orthoprojector. In the case of boundedness of  $\Omega$  (further this requirement will be assumed as the default setting) we uniquely continue the orthoprojector  $\Pi$  to the projector on  $L_q(\Omega; \mathbb{R}^n)$ , which we denote by the same symbol  $\Pi : L_q(\Omega; \mathbb{R}^n) \rightarrow L_q(\Omega_0; \mathbb{R}^n)$ ,  $q \in [1, 2]$ .

**Definition 1.** Let  $\xi \in L_1(\Omega; \mathbb{R}^n)$ . The random value  $\Pi\xi \in L_1(\Omega_0; \mathbb{R}^n)$  is called the *conditional mathematic expectation*  $\xi$  relatively to  $\mathcal{A}_0$  and is denoted by  $E(\xi|\mathcal{A}_0)$ .

Note that if  $E(\xi|\mathcal{A}) = \xi$  and  $E(\xi|\mathcal{A}_0)$  is mathematic expectation, if  $\mathcal{A}_0 = \{\emptyset, \Omega\}$ . Let's also note that  $E(\xi|\mathcal{A}_0)$  is the unique (up to values on the set of zero probability) random variable from

$L_1(\Omega; \mathbb{R}^n)$  such that  $\int_{\mathcal{A}} \xi d\mathbf{P} = \int_{\mathcal{A}} E(\xi|\mathcal{A}_0)d\mathbf{P}$  for any  $\mathcal{A} \in \mathfrak{F}_0$ . This consequence of the Radon – Nikodym theorem can be considered as the equivalent definition  $E(\xi|\mathcal{A}_0)$ .

Let  $\mathcal{J}_a^b \subset \mathbb{R}$  be some interval,  $-\infty \leq a < b \leq +\infty$ . Consider the following mappings:  $f : \mathcal{J}_a^b \rightarrow \mathcal{V}(\Omega; \mathbb{R}^n)$ , for each  $t \in \mathcal{J}_a^b$  putting in correspondence  $\xi \in \mathcal{V}(\Omega; \mathbb{R}^n)$  and  $g : \mathcal{V}(\Omega; \mathbb{R}^n) \times \Omega \rightarrow \mathbb{R}^n$ , for each pair  $(\xi, \omega)$  putting in correspondence  $\xi(\omega) \in \mathbb{R}^n$ . Under the *random process* we call the mapping  $\eta : \mathcal{J}_a^b \times \Omega \rightarrow \mathbb{R}^n$  which has the form  $\eta = \eta(t, \omega) = g(f(t), \omega)$ . Thus, at each fixed  $t \in \mathcal{J}_a^b$  the random process  $\eta = \eta(t, \cdot)$  is the random variable, i.e.  $\eta(t, \cdot) \in \mathcal{V}(\Omega; \mathbb{R}^n)$ , and at each fixed  $\omega \in \Omega$  the random process  $\eta = \eta(\cdot, \omega)$  is called as the (*sample*) *path*. We denote set of random processes by symbol  $\mathcal{P}(\mathcal{J}_a^b \times \Omega; \mathbb{R}^n)$ .

With each  $\xi \in \mathcal{V}(\Omega; \mathbb{R}^n)$  we connect  $\sigma$ -algebra  $\mathcal{A}^\xi \subset \mathcal{A}$ , i.e. minimal  $\sigma$ -subalgebra  $\mathcal{A}$ , relatively to which  $\xi$  is measurable.  $\mathcal{A}^\xi$  is called  $\sigma$ -algebra, generated by  $\xi$ . The equivalent definition of  $\mathcal{A}^\xi$  is that it's minimal  $\sigma$ -algebra, containing prototypes of all Borel sets in  $\mathbb{R}^n$  under mapping  $\xi : \Omega \rightarrow \mathbb{R}^n$ . With each  $\eta \in \mathcal{P}(\mathcal{J}_a^b \times \Omega; \mathbb{R}^n)$  we connect three families of  $\sigma$ -subalgebras of the  $\sigma$ -algebra  $\mathcal{A}$ :

- the past  $\mathfrak{P}_t^\eta$ , generated by random variables  $\eta(S, \cdot)$  at each  $S \in (a, t)$ ;
- the future  $\mathfrak{F}_t^\eta$ , generated by random variables  $\eta(S, \cdot)$  at each  $S \in (t, b)$ ;
- the present  $\mathfrak{N}_t^\eta$ , generated by the random variable  $\eta(t, \cdot)$ .

All  $\sigma$ -algebras we consider complete, i.e. containing sets of zero probability.

Let's rename  $L_q(\Omega) \equiv L_q(\Omega; \mathbb{R}^n)$  and by  $L_q(\mathcal{J}_a^b \times \Omega)$  we denote the set  $\{\eta \in \mathcal{P}(\mathcal{J}_a^b \times \Omega; \mathbb{R}^n) : \eta(t, \cdot) \in L_q(\Omega)$  at all  $t \in \mathcal{J}_a^b\}$ ,  $q \in [1, +\infty)$ . Remind that  $\eta \in \mathcal{P}(\mathcal{J}_a^b \times \Omega; \mathbb{R}^n)$  is called *random process with a.s. continuous paths*, if for  $\mathbf{P}$ -a.s. all  $\omega \in \Omega$  paths  $\eta(\cdot, \omega)$  are continuous. The set of random processes from the  $L_q(\mathcal{J}_a^b \times \Omega)$ , whose paths are a.s. continuous, is denoted by  $L_q^0(\mathcal{J}_a^b \times \Omega)$ . For short, we also rename  $E_t^\eta = E(\eta|\mathfrak{N}_t^\eta)$ .

**Definition 2.** Let  $\eta \in L_q^0(\mathcal{J}_a^b \times \Omega)$ ,  $q \in [1, 2]$ , the mean right  $D\eta(t, \cdot)$  (left  $D_*\eta(t, \cdot)$ ) derivative of the random process  $\eta$  at the point  $t \in \mathcal{J}_a^b$  is called as the random variable

$$D\eta(t, \cdot) = \lim_{\Delta t \rightarrow 0+} E_t^\eta \left( \frac{\eta(t + \Delta t, \cdot) - \eta(t, \cdot)}{\Delta t} \right)$$

$$\left( D_*\eta(t, \cdot) = \lim_{\Delta t \rightarrow 0-} E_t^\eta \left( \frac{\eta(t, \cdot) - \eta(t - \Delta t, \cdot)}{\Delta t} \right) \right),$$

if the limit exists in the sense of the uniform metric on  $\mathbb{R}$ . The random process  $\eta$  is called *mean right (left) differentiable on the  $\mathcal{J}_a^b$* , if in each point  $t \in \mathcal{J}_a^b$  the mean right (left) derivative exists.

So, let  $\eta \in L_q^0(\mathcal{J}_a^b \times \Omega)$ ,  $q \in [1, 2]$ , be mean right (left) differentiable on  $\mathcal{J}_a^b$ . Its mean right (left) derivative is also the random process, which we denote by the symbol  $D\eta$  ( $D_*\eta$ ). If  $\eta \in L_q^0(\mathcal{J}_a^b \times \Omega)$ ,  $q \in [1, 2]$  is mean right and left differentiable on  $\mathcal{J}_a^b$ , we can determine the symmetrical (antisymmetrical) mean derivative  $D_S\eta = \frac{1}{2}(D + D_*)\eta$  ( $D_A\eta = \frac{1}{2}(D - D_*)\eta$ ). Further for the sake of brevity symmetrical mean derivative  $D_S$  will be called the *Nelson – Gliklikh derivative*. Note that if all random variables  $\eta(t, \cdot)$ ,  $t \in \mathcal{J}_a^b$ , of the process  $\eta \in L_1^0(\mathcal{J}_a^b \times \Omega)$  take values with the probability 1, and all paths  $\eta(\cdot, \omega)$  are a.s. differentiable, the Nelson – Gliklikh derivative coincides with the partial derivative with respect to the first argument.

Consider the Wiener processes and their Nelson – Gliklikh derivative. Assume  $\mathcal{J}_a^b = \mathbb{R}_+$  ( $= [0, +\infty)$ ) and consider the family  $\mathcal{B} = \{\mathfrak{B}_t \subset \mathcal{A} : t \in \mathbb{R}_+\}$  of  $\sigma$ -subalgebras of  $\sigma$ -algebra  $\mathcal{A}$  on  $\Omega$ . Assume the family  $\mathcal{B}$  be *never-decreasing*, i.e.  $\mathfrak{B}_s \supset \mathfrak{B}_t$  for all  $s \geq t$ . The process  $\eta \in \mathcal{P}(\mathbb{R}_+ \times \Omega; \mathbb{R}^n)$  is called the *martingale with respect to family  $\mathcal{B}$* , if for any  $t \geq s$  we have the equality  $E(\eta(t, \cdot)|\mathfrak{B}_s) = \eta(s, \cdot)$ .

**Definition 3.** The random process  $w \in L_2^0(\mathbb{R}_+ \times \Omega)$  is called the *Wiener process*, subaltern to  $\mathcal{B}$ , if the Wiener process is the martingale with respect to  $\mathcal{B}$ ,  $w(0, \cdot) = 0$  and  $E((w(t, \cdot) - w(s, \cdot))^2 | \mathfrak{B}_t) = t - s$  for  $t \geq s$ .

Let's formulate some interesting and useful consequence of Definition 3 as properties of the Wiener process.

(Cw1) (B. Levi) the Wiener process  $w$  has stationary independent Gaussian increments, moreover  $E(w(t, \cdot) - w(s, \cdot)) = 0$  and  $E((w(t, \cdot) - w(s, \cdot))^2) = |t - s|$ .

In other words, the increment  $w(t, \cdot) - w(s, \cdot)$  of the Wiener process does not depend on family  $\mathcal{B}$  and has the same (Gaussian) distribution, as the process  $w(t - s, \cdot)$  for  $t \geq s$ . The same can be said about the Wiener process.

(Cw2) Paths  $w(\cdot, \omega)$  of the Wiener process are a.s. not differentiable at any point  $t \in \mathbb{R}_+$  and have unbounded variation on any interval  $\mathfrak{I}_a^b \subset \mathbb{R}_+$ .

As mentioned, above this property is the main obstacle in the mathematical research of the WN. Now list the properties of the Wiener process, which will be useful in the investigation of the «WN».

**Theorem 1.** (J.E. Glikhlih) (i)  $D_S w(t, \cdot) = (2t)^{-1} w(t, \cdot)$ ,  $D_S^2 w(t, \cdot) = -(2t^2)^{-1} w(t, \cdot)$ ;

$$(ii) \int_0^t \|D_S w(s)\| ds < +\infty.$$

Here  $\|\cdot\|$  – is the norm in  $L_1(\Omega)$ . Define  $L_q^k((0, \tau) \times \Omega) = \{\eta \in L_q^{k-1}((0, \tau) \times \Omega) : \text{there exist } D_S^k \eta, \text{ and all its paths are a.s. continuous on } (0, \tau) \text{ for any } \tau > 0\}$ ,  $k \in \mathbb{N}$ ,  $q \in [1, +\infty)$ . From the theorem 1 (i) follows that  $w \in L_q^2((0, \tau) \times \Omega)$ ,  $q \in [1, 2]$ . There is stronger result:

**Theorem 2.** (Yu.E. Glikhlih)  $w \in L_q^k((0, \tau) \times \Omega)$ ,  $k \in \mathbb{N}$ ,  $q \in [1, 2]$ .

## 2. The Weakened Showalter – Sidorov problem for Leontieff type equations with the additive «white noise»

Let's start with the simplification of the definitions. First of all, we rename  $\mathcal{P}(\mathfrak{I}_a^b \times \Omega) \equiv \mathcal{P}(\mathfrak{I}_a^b \times \Omega; \mathbb{R}^n)$  the set of random processes with the values in  $\mathbb{R}^n$ . The random process  $\eta \in \mathcal{P}(\mathfrak{I}_a^b \times \Omega)$  will be denoted by symbols  $\eta = \eta(t)$ , considering its dependence on the second variable  $\omega \in \Omega$  taking place by default. Nelson – Glikhlih derivative  $D_S$  (if it exists) of the random process  $\eta$  will be denoted by symbol  $\overset{\circ}{\eta}$ , i.e.  $D_S \eta = D_S \eta(t) = \overset{\circ}{\eta} = \overset{\circ}{\eta}(t)$ . The subset (the Lebesgue space) of random processes  $\mathcal{P}(\mathfrak{I}_a^b \times \Omega)$ , with a.s. continuously differentiable up to order  $k$  inclusive (in the sense of Nelson – Glikhlih) paths, denote by  $L_q^k(\mathfrak{I}_a^b \times \Omega)$ ,  $k \in \{0\} \cup \mathbb{N}$ .

Let further  $L$  и  $M$  be square matrixes of order  $n$ ,  $\det L = 0$ , and the pencil  $\mu L - M$  be  $p$ -regular [2]. Consider Leontieff type equations

$$L \overset{\circ}{\eta} = M \eta + \overset{\circ}{w}, \tag{5}$$

where  $\overset{\circ}{w} = (2t)^{-1} w(t) - \ll \text{WN} \gg$ . The random process  $\eta \in L_1^1((0, \tau) \times \Omega)$  is called a *solution of the equation (5)*, if a.s. all its paths satisfy (5) at any  $\tau \in (0, +\infty)$ . A weakened (in the sense of S.G. Krein) Showalter – Sidorov problem we call the following problem

$$\lim_{t \rightarrow 0+} [R_\alpha^L(M)]^{p+1} (\eta(t) - \xi_0) = 0, \tag{6}$$

where  $R_\alpha^L(M) = (\alpha L - M)^{-1} L$  is the right  $L$ -resolvent of the matrix  $M$ ,  $\alpha \in \rho^L(M)$  –  $L$ -resolvent set of the matrix  $M$ . We note immediately that since  $\ker [R_\alpha^L(M)]^{p+1} = \ker P$ ,  $\text{im} [R_\alpha^L(M)]^{p+1} =$

$\text{im}P$ , where

$$P = \frac{1}{2\pi i} \int_{\gamma} R_{\mu}^L(M) d\mu$$

is projector on  $\mathbb{R}^n$ , so the problem (6) is equivalent to the problem

$$\lim_{t \rightarrow 0+} P(\eta(t) - \xi_0) = 0. \tag{7}$$

So, the solution  $\eta = \eta(t)$  of the equations (5) will be called the *solution of the problem (5), (6)* (or, that is also equivalent, – (5), (7)), if it a.s. satisfies (6).

Along with the projector  $P$  we introduce for consideration the projector

$$Q = \frac{1}{2\pi i} \int_{\gamma} L_{\mu}^L(M) d\mu,$$

where  $L_{\mu}^L(M) = L(\mu L - M)^{-1}$  is the left  $L$ -resolvent of the matrix  $M$ , and  $\gamma \in \mathbb{C}$ , as above, is the closed contour, bounding the  $L$ -spectrum  $\sigma^L(M)$  of the matrix  $M$ .

**Lemma 1.** *Let the pencil  $\mu L - M$  be  $p$ -regular, then  $\dim \ker P = \dim \ker Q$ ,  $LP = QL$ ,  $MP = QM$ .*

In view of  $p$ -regularity of the pencil  $\mu L - M$  without loss of generality, we can assume  $\det M \neq 0$ . Indeed, by making in (5) replacement  $\eta(t) = \omega(t)e^{\alpha t}$ , where  $\alpha \in \rho^L(M)$ , we will reach of the form (5) at a system, and in the right side will be the matrix  $M' = M - \alpha L$ . Obviously,  $\det M' \neq 0$ . Let's construct the matrix  $(\mathbb{I}_n - P)M^{-1}(\mathbb{I}_n - Q)L(\mathbb{I}_n - P) \equiv H$ .

**Lemma 2.** *Let the pencil  $\mu L - M$  be  $p$ -regular,  $\det M \neq 0$ , then the matrix  $H$  is nilpotent with degree not above  $p$ .*

**Lemma 3.** *Let the pencil  $\mu L - M$  be  $p$ -regular, then there exists the square matrix  $\Lambda$  of order  $n$  and  $\Lambda QL = LPA = \text{diag}\{\mathbb{O}_m, \mathbb{I}_l\}$  is fulfilled up to permutations of rows, where  $m = \dim \ker P$ ,  $l = n - m$ .*

Assume  $S = \Lambda QM$  and construct

$$e^{tS} = \frac{1}{2\pi i} \int_{\gamma} R_{\mu}^L(M) e^{\mu t} d\mu.$$

It is easy to see that the family  $\{e^{tS} : t \in \mathbb{R}\}$  forms a group, and its unit  $e^{tS}|_{t=0} = P$ .

**Theorem 3.** *Let the pencil  $\mu L - M$  be  $p$ -regular,  $\det M \neq 0$ . Then for any  $\xi_0 \in \mathcal{V}(\Omega; \mathbb{R}^n)$  and  $\tau \in (0, +\infty)$  there exists the solution  $\eta \in L_1^1((0, \tau) \times \Omega)$  of the problem (5), (6), and all solutions a.s. have the form*

$$\eta(t) = - \sum_{k=0}^p H^k M^{-1}(\mathbb{I}_n - Q) D_S^k \overset{\circ}{w}(t) + e^{tS} \xi_0 + \int_0^t e^{(t-s)S} \Lambda Q \overset{\circ}{w}(s) ds. \tag{8}$$

Here  $H^0 = \mathbb{I}_n - P$  by construction. From theorem 1 (ii) and theorem 2 it follows that  $\eta \in L_1^1((0, \tau) \times \Omega)$  for all  $\tau \in (0, +\infty)$ . By direct substitution we see that (8) is a solution of (5). Similarly [9], Chapter 4, we see that there are a.s. no solutions of (5) different from (8). Since by action of the projector  $P$  one the first term (in fact, subtrahend) in (8), we obtain identically zero at all  $t \in (0, \tau)$ , it is clear that (7), and thus (6) are also satisfied.

**Example 1.** One of the simplest models of the MT, considered in [6], [7], has the following form (applied to the situation considered here)

$$\overset{\circ}{\alpha} = A\alpha + \overset{\circ}{\omega}, \quad \beta = C\alpha. \tag{9}$$

Here the random process  $\alpha = \alpha(t)$  simulates condition of the MT, matrixes  $A$  и  $C$  – its devices,  $\beta = \beta(t)$  is the observed process, in this case, «WN»  $\overset{\circ}{\omega} = \overset{\circ}{\omega}(t)$ . Assuming  $\eta = (\alpha_1, \dots, \alpha_m, \beta_1, \dots, \beta_l)$ ,  $L = \text{diag}\{\mathbb{I}_l, \mathbb{O}_m\}$ ,

$$M = \begin{pmatrix} A & \mathbb{O} \\ C & -\mathbb{I}_m \end{pmatrix},$$

we will reach the system (5)  $\overset{\circ}{w} = (\overset{\circ}{\omega}_1, \dots, \overset{\circ}{\omega}_l, \underbrace{0, \dots, 0}_l)$ . Firstly, as it is easy to see  $\det M \neq 0$ , since  $\det M = (-1)^m \det A$  and  $\det A \neq 0$  by construction. And secondly, the pencil  $\mu L - M$  is 0-regular. Projectors  $Q = L$ ,

$$P = \begin{pmatrix} \mathbb{I}_l & \mathbb{O} \\ C & \mathbb{O}_m \end{pmatrix},$$

operator  $\Lambda = L$ , the group of solution operators

$$e^{tM} = \begin{pmatrix} e^{tA} & \mathbb{O} \\ Ce^{tA} & \mathbb{O}_m \end{pmatrix},$$

where  $e^{tA} = \frac{1}{2\pi i} \int_{\gamma} (\mu \mathbb{I}_l - A)^{-1} e^{\mu t} d\mu$ , and the contour  $\gamma$  limits the region containing the spectrum  $\sigma(A)$  of the matrix  $A$  (Incidentally, here  $\sigma(A) = \sigma^L(M)$ ). In the model (9) we are interested just in observation  $\beta$ , and in addition, for technical reasons, the initial random variable  $\xi_0$  (in (6), (7)) can be set equal to zero a.s. Therefore, by Theorem 3 is valid

**Corollary 1.** *All observations in the model (9) a.s. are given by formula*

$$\beta(t) = C \int_0^t e^{(t-s)A} \overset{\circ}{\omega}(s) ds, \quad t \in \mathbb{R}_+.$$

This raises two important conclusions: firstly, the random process of observation  $\beta(0) = 0$  is a.s., and secondly, paths  $\beta(t)$  are a.s. continuous.

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## Об измерении «белого шума»

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В рамках теории уравнений леонтьевского типа рассмотрена математическая модель измерительного устройства, демонстрирующая эффект механической инерционности. При изучении модели с детерминированным внешним сигналом очень полезными оказались методы и результаты теории уравнений соболевского типа и вырожденных групп операторов, поскольку они позволили создать эффективный вычислительный алгоритм. Теперь в модели предполагается наряду с детерминированным сигналом наличие белого шума. Поскольку модель представлена вырожденной системой обыкновенных дифференциальных уравнений, то к ней трудно применимы существующие ныне подходы Ито – Стратоновича – Скорохода и Мельниковой – Филинкова – Альшанского, в которых белый шум понимается как обобщенная производная винеровского процесса. Вместо этого предлагается новая концепция «белого шума», равной симметрической производной в среднем (в статье – производной Нельсона – Гликлиха) винеровского процесса, причем подмечено, что в рамках теории Эйнштейна – Смолуховского данная производная совпадает с «обычной» производной броуновского движения. В первой части статьи собраны основные факты теории производной Нельсона – Гликлиха, адаптированные к рассматриваемой ситуации. Во второй – рассмотрена ослабленная задача Шоултера – Сидорова и даны точные формулы ее решения. В качестве примера приведена конкретная модель измерительного устройства.

*Ключевые слова:* уравнения леонтьевского типа, ослабленная задача Шоултера – Сидорова, симметрическая производная в среднем, винеровский процесс.

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