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INVERSE PROBLEMS FOR MATHEMATICAL MODELS OF QUASISTATIONARY ELECTROMAGNETIC WAVES IN ANISOTROPIC NONMETALLIC MEDIA WITH DISPERSION

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We consider inverse problems of evolution type for mathematical models of quasistationary electromagnetic waves. It is assumed in the model that the wave length is small as compared with space inhomogeneities. In this case the electric and magnetic potential satisfy elliptic equations of second order in the space variables comprising integral summands of convolution type in time. After differentiation with respect to time the equation is reduced to a composite type equation with an integral summand. The boundary conditions are supplemented with the overdetermination conditions which are a collection of functionals of a solution (integrals of a solution with weight, the values of a solution at separate points, etc.). The unknowns are a solution to the equation and unknown coefficients in the integral operator. Global (in time) existence and uniqueness theorems of this problem and stability estimates are established.

Keywords: Sobolev-type equation; equation with memory; elliptic equation; inverse problem; boundary value problem.

Introduction

We consider the problems arising in the description of propagation of both electromagnetic waves in anisotropic media [1] and nonstationary interior waves in an incompressible stratified rotating fluid [2]. The peculiarities of propagation of electromagnetic waves in anisotropic media are defined by the corresponding material equations. If the length of a wave is small as compared with space inhomogeneities that these equations can be written in the form accounting for time dispersion only and introducing the potentials of electric and magnetic fields $E = -\nabla\varphi(x,t)$ and $H = -\nabla\psi(x,t)$ and making some transformations we arrive at the equations (see [1], p. 28)

$$\sum_{i=1}^{3}(1+4\pi\kappa_i*)\varphi_{x_i,x_i} = -4\pi\rho + F_d, \quad \sum_{i=1}^{3}(1+4\pi\kappa_i*)\psi_{x_i,x_i} = F_b, \tag{1}$$

where κ_i are the diagonal entries of the tensors of electric and magnetic susceptibilities and $\kappa_i*\varphi(x,t) = \int_0^t \kappa_i(t-\tau)\varphi(x,\tau)\,d\tau$. Note that some model problems for nonstationary waves in media with anisotropic dispersion are reduced to integro-differential equations (1) with kernels of convolution operators of the form of a sine, a polynomial, or an exponential function. In these cases it is possible to reduce an initial vector systems of equations by introducing generalized potentials of quasistationary electric and magnetic fields to composite type equations (see [2]) of the form

$$P_s(\partial_t)\Delta\Phi(x,t) + P_{mij}(\partial_t)\sum_{i,j=1}^{3}\Phi_{x_ix_j} = F,$$

where P_s, P_{mij} are polynomials of degrees s and m, respectively. At the present article we examine inverse problems on recovering the coefficients k_i for general equations of the form

$$L_0 u + \sum_{i=1}^{m} \kappa_i * L_i u = f, \tag{2}$$

where

$$L_k u = \sum_{i,j=1}^{n} a_{ij}^k(x,t) u_{x_i,x_j} + \sum_{i=1}^{n} a_i^k(x,t) + a_0^k(x,t)u, \quad (x,t) \in Q = G \times (0,T), \ G \subset \mathbb{R}^n.$$

The equation (2) is supplemented with the overdetermination conditions

$$\Psi_j(u)(t) = \psi_j(t), \tag{3}$$

where Ψ_i are some functionals (the conditions on them are described below), and the boundary conditions

$$Bu|_S = g(x,t), \ S = \partial G \times (0,T), \tag{4}$$

where $Bu = u$ or $Bu = \sum_{i=1}^{n} \gamma_i(x,t) u_{x_i} + \sigma(x,t)u$. Similar equations and systems of equations arise in elasticity (materials with memory) [3–5], physics (phase-field models, heat and mass transfer) [6, 7], and in many other fields. The most known case is the case of a parabolic (see [3, 6–12]) or hyperbolic (see [4, 5]) operator L_0. Even the most general case was studied in which $L_0 = \partial_t - A$ or $L_0 = \partial_t^2 - A$, with A a generator of an analytic semigroup (see, for instance, [9–12]). The case of a pseudoparabolic operator L_0 is treated in [13]. In the case of $L_0 = \partial_t$, we arrive at Gurtin–Pipkin-type models (see [14, 15]). Probably, the elliptic case was not considered except for one model situation (see [16]), where $n = 1$. We establish global (in time) solvability of the problem (2) – (4) in Sobolev spaces.

1. Preliminaries

We employ the Sobolev spaces $W_p^s(G)$ and Hölder spaces $C^\alpha(\overline{G})$. The symbol $L_p(0,T;H)$ (H is a Banach space) stands for the spaces of strongly measurable functions defined on $[0,T]$ with values in H (see the definition of the function spaces, for instance, in [17]).

We assume below that $\Gamma = \partial G \in C^2$ (see the definition, for example [18, Sect. 1, Ch. 1]) and that the coefficients of the operators L_k ($k = 0, 1, \ldots, m$) are real-valued and the operator L_0 is elliptic, i. e., there exists a constant $\delta_0 > 0$ such that

$$\sum_{i,j=1}^{m} a_{ij}^0 \xi_i \xi_j \geq \delta_0 |\xi|^2 \quad \forall \ \xi \in R^n, \quad \forall \ (x,t) \in \overline{Q}.$$

We fix the parameter $p > n$ (for simplicity) and suppose that

$$\begin{gathered} a_{ij}^0 \in C(\overline{Q}), \quad a_i^0, a_0^0 \in C([0,T]; L_p(G)), \ a_{ij}^k \in L_\infty(Q), \ a_{ijt}^0, a_{ijt}^k \in L_p(0,T; L_\infty(G)), \\ a_i^k, a_0^k \in L_\infty(0,T; L_p(G)), \ a_{0t}^k, a_{it}^k \in L_p(Q) \quad (i,j = 1,2,..,n, \ k = 1,2,\ldots,m), \\ \gamma_i, \gamma_{it}, \sigma, \sigma_t \in C^1(\overline{S}) \ (i = 1,\ldots,n), \quad |\textstyle\sum_{i=1}^{n} \gamma_i n_i| \geq \delta_1 > 0 \ \forall (x,t) \in S, \end{gathered} \tag{5}$$

where $\vec{n} = (n_1, n_2, \ldots, n_n)$ is the outward unit normal to S and δ_1 is a constant. The operator L_0 is assumed invertible, i. e., the following theorem is valid.

Theorem 1. *Let the condition* (5) *hold. The problem with a parameter*

$$L_0(t)u = f(x), \qquad B(t)u|_\Gamma = g(x), \tag{6}$$

for every $f \in L_p(G)$ $(p > n)$ and $g \in W_p^{s_0}(\Gamma)$ ($s_0 = 2 - 1/p$ in the case of the Dirichlet conditions and $s_0 = 1 - 1/p$ the case of the oblique derivative problem) has a unique solution $u \in W_p^2(G)$ satisfying the estimate

$$\|u\|_{W_p^2(G)} \leq c(\|f\|_{L_p(G)} + \|g\|_{W_p^{s_0}(\Gamma)}),$$

where the constant c is independent of $f, g, t \in [0, T]$.

The claim of the theorem holds whenever $ker\, L_0 = 0$. In particular, it suffices to require in the case of the Dirichlet boundary conditions that $a_0^0 \leq 0$ a.e. in Q (see the maximum principle [19, Ch. 8]) and in the case of the oblique derivative problem that $a_0^0 \leq 0$ a.e. in Q and $a_0^0 < 0$ a.e. in some neighborhood about S (see Proposition 2.3.2 and Theorem 2.3.5 in [20]).

Corollary 1. *As a direct corollary of the claim of the theorem, we have that if $f(x,t) \in C([\alpha, \beta]; L_p(G))$ and $g(x,t) \in C([\alpha, \beta]; W_p^{s_0}(\Gamma))$ $(0 \leq \alpha < \beta \leq T)$ then the problem*

$$L_0(t)u = f(x,t), \qquad Bu|_\Gamma = g(x,t), \tag{7}$$

has a unique solution $u \in C([\alpha, \beta]; W_p^2(G))$ satisfying the estimate

$$\|u\|_{C([\alpha,\beta];W_p^2(G))} \leq c(\|f\|_{C([\alpha,\beta];L_p(G))} + \|g\|_{C([\alpha,\beta];W_p^{s_0}(\Gamma))}),$$

where the constant c is independent of f, g, α, β. It is not also difficult to demonstrate that if there exist the generalized derivatives $f_t \in L_p(\alpha, \beta; L_p(G))$, $g_t \in L_p(\alpha, \beta; W_p^{s_0}(\Gamma))$ then a solution to the problem (7) *is differentiable with respect to t, $u_t \in L_p(\alpha, \beta; W_p^2(G))$ and*

$$L_0(t)u_t = f_t(x,t) - L_{0t}u, \qquad Bu_t|_\Gamma = g_t(x,t) - B_t u|_\Gamma, \tag{8}$$

for almost all $t \in (\alpha, \beta)$ (the coefficients of the operators L_{0t}, B_t are the derivatives with respect to t of the coefficients of L_0, B). Moreover, the following estimate is valid:

$$\|u_t\|_{L_p(\alpha,\beta;W_p^2(G))} + \|u\|_{C([\alpha,\beta];W_p^2(G))} \leq$$
$$\leq c(\|f\|_{C([\alpha,\beta];L_p(G))} + \|g\|_{C([\alpha,\beta];W_p^{s_0}(\Gamma))} + \|f_t\|_{L_p(\alpha,\beta;L_p(G))} + \|g_t\|_{L_p(\alpha,\beta;W_p^{s_0}(\Gamma))}),$$

where the constant c is independent of α, β, f, g.

Note that the conditions (5) imply that the coefficients a_{ij}^k, a_i^k, a_0^k belong to the space $C([0, T]; L_p(G))$ after a possible modification on a set of zero measure. In what follows we assume this condition to be fulfilled.

Lemma 1. *The following inequalities hold:*

$$\|u * v\|_{L_p(0,\gamma)} \leq \|u\|_{L_p(0,\gamma)}\|v\|_{L_1(0,\gamma)}, \|u * v\|_{L_p(0,\gamma)} \leq \gamma^{1-1/p}\|u\|_{L_p(0,\gamma)}\|v\|_{L_p(0,\gamma)}, \tag{9}$$

$$\|u(t)\|_{L_\infty(0,\gamma)} \le \gamma^{1-1/p}\|u_t\|_{L_p(0,\gamma)}, \quad u(0)=0, \tag{10}$$

$$\|u(t)\|_{L_p(0,\gamma)} \le \gamma\|u_t\|_{L_p(0,\gamma)}, \quad u(0)=0. \tag{11}$$

The inequalities (9) are known (see, for instance, Lemma 3.1 in [6]). The inequalities (10), (11) are an obvious consequence of the Newton–Leibnitz formula.

Assume that a solution to the problem (2) – (4) possesses the property $u_t \in L_p(0,T;W_p^2(G))$ and $u \in C([0,T];W_p^2(G))$, the conditions (5) hold, $g \in C([0,T];W_p^{s_0}(\Gamma))$, and $g_t \in L_p(0,T;W_p^{s_0}(\Gamma))$, $k_i(t) \in L_p(0,T)$ for all i, and $f, f_t \in L_p(Q)$. Taking $t=0$ in (2), we infer $L_0(x,0)u(x,0) = f(x,0)$. Applying Theorem 1, we can find the function $u(x,0) = u_0(x)$ as a solution to the problem (6) at $t=0$. The boundary condition (3) yields

$$Bu_0(x)|_\Gamma = g(x,0). \tag{12}$$

The condition (4) implies the necessary solvability condition

$$\Psi_j(u_0(x)) = \psi_j(0) \quad j=1,2,\ldots,m. \tag{13}$$

Next, differentiating (2) with respect to t, we infer

$$(L_0u)_t + \int_0^t \sum_{i=1}^m k_i(\tau)(L_iu_t(x,t-\tau) + L_{it}u(t-\tau))\,d\tau = f_t - \sum_{i=1}^m k_i(t)L_i(x,0)u_0(x). \tag{14}$$

Construct an auxiliary function $\Phi(x,t) \in C([0,T];W_p^2(G))$ such that $\Phi_t \in L_p(0,T;W_p^2(G))$, $B\Phi|_S = g(x,t)$, and $\Phi(x,0) = u_0(x)$. Let Φ be a solution to the problem (7). Making the change of variables $u = v + \Phi$, we obtain the problem

$$\begin{gathered}(L_0v)_t + \int\limits_0^t \sum_{i=1}^m k_i(\tau)(L_iv)_t(x,t-\tau)\,d\tau = \\ = -\sum_{i=1}^m k_i(t)L_i(x,0)u_0(x) - \int\limits_0^t \sum_{i=1}^m k_i(\tau)(L_i\Phi)_t(x,t-\tau)\,d\tau = f_0,\end{gathered} \tag{15}$$

$$Bv|_\Gamma = 0, \quad v(x,0) = 0, \tag{16}$$

$$\Psi_j(v) = \psi_j - \Psi_j(\Phi) = \tilde{\psi}_j, \; j=1,2,\ldots,m. \tag{17}$$

Theorem 2. *Assume that $f, f_t \in L_p(Q)$, the conditions (5), (12), (13) hold, $g \in C([0,T];W_p^{s_0}(\Gamma))$, $g_t \in L_p(0,T;W_p^{s_0}(\Gamma))$ $(p > n)$, and $\psi_j \in W_p^1(0,T)$ $(j=1,2,\ldots,m)$. Then the problem (15) – (17) of determining the functions $v, k_1, \ldots, k_m$ from the class $v_t \in L_p(0,T;W_p^2(G))$, $v \in C([0,T];W_p^2(G))$, $k_i \in L_p(0,T)$ $(i=1,2,\ldots,m)$ is equivalent to the problem (2) – (4) of determining the functions $u, k_1, \ldots, k_m$ such that $u_t \in L_p(0,T;W_p^2(G))$, $u \in C([0,T];W_p^2(G))$, and $k_i \in L_p(0,T)$ $(i=1,2,\ldots,m)$.*

Proof. Actually, the arguments presented before the theorem show that if $u, k_1, \ldots, k_m$ is a solution to the problem (2) – (4) from the above–pointed class then the functions $v, k_1, \ldots, k_m$ is a solution to the problem (15) – (17). So it suffices to verify the converse statement. Let $v, k_1, \ldots, k_m$ is a solution to the problem (15) – (17). Put $u = v + \Phi$. The equalities $f_t = (L_0\Phi)_t$ and (15) imply that

$$\partial_t(L_0u) + \partial_t \int\limits_0^t \sum_{i=1}^m k_i(\tau)L_iu(x,t-\tau)\,d\tau = f_t, \tag{18}$$

Integrating this equality from 0 to t, we derive that

$$L_0 u + \int\limits_0^t \sum_{i=1}^m k_i(\tau) L_i u(x, t-\tau)\, d\tau = f(t) + L_0 u(x,0) - f(x,0).$$

Since $0 = v(x,0) = u(x,0) - \Phi(x,0) = u(x,0) - u_0(x)$, $u(x,0) = u_0(x)$, the definition of the function u_0, yields $Lu(x,0) - f(x,0) = 0$, i. e., the equality (2) holds. The validity of the condition (4) is obvious.

□

Let $0 \le \alpha < \beta \le T$. Define the space $H(\alpha,\beta)$ as the space of functions $v(x,t)$ such that $v_t \in L_p(\alpha,\beta; W_p^2(G))$, $v \in C([\alpha,\beta]; W_p^2(G))$, $Bv|_\Gamma = 0$. Endow it with the norm $\|v|_{H(\alpha,\beta)} = \|v\|_{C([\alpha,\beta];W_p^2(G))} + \|v_t\|_{L_p(\alpha,\beta;W_p^2(G))}$. In what follows, a norm of a vector is the sum of the norms of its coordinates.

Lemma 2. *Let the conditions* (5) *hold. Then*

$$\|(L_j v)_t\|_{L_p(\alpha,\beta;L_p(G))} \le c\|v\|_{H(\alpha,\beta)} \quad \forall v \in C([\alpha,\beta]; W_p^2(G)) : \ v_t \in L_p(\alpha,\beta; W_p^2(G)), \quad (19)$$

$$\|L_{jt} v\|_{L_p(\alpha,\beta;L_p(G))} \le c\|v\|_{C([\alpha,\beta];W_p^2(G))} \forall v \in C([\alpha,\beta]; W_p^2(G)) : v_t \in L_p(\alpha,\beta; W_p^2(G)), \quad (20)$$

where $0 \le \alpha < \beta \le T$ *and the constant* c *is independent of* α, β, j.

Proof. The proof is more or less obvious. As an example, we establish (20). The expression $L_{jt} v$ contains the summands $a^j_{ikt} v_{x_i x_k}$, $a^j_{kt} v_{x_k}$, and $a^j_{0t} v$. We have

$$\|a^j_{ikt} v_{x_i x_k}\|_{L_p(\alpha,\beta;L_p(G))} \le \Big(\int\limits_\alpha^\beta \|a^j_{ikt}\|^p_{L_\infty(G)}\, dt\Big)^{1/p} \|v_{x_i x_k}\|_{L_\infty(\alpha,\beta;L_p(G))} \le c^j_{ik}(\alpha,\beta)\|v\|_{H(\alpha,\beta)}. \quad (21)$$

Similarly, we infer

$$\|a^j_{kt} v_{x_k}\|_{L_p(\alpha,\beta;L_p(G))} \le \|a^j_{kt}\|_{L_p(\alpha,\beta;L_p(G))} \|v_{x_k}\|_{L_\infty(\alpha,\beta;L_\infty(G))} \le c^j_k(\alpha,\beta)\|v\|_{H(\alpha,\beta)}, \quad (22)$$

$$\|a^j_{0t} v\|_{L_p(\alpha,\beta;L_p(G))} \le c^j_0(\alpha,\beta)\|v\|_{H(\alpha,\beta)}. \quad (23)$$

Here we employ the embedding $W_p^1(G) \subset L_\infty(G)$ (see [17]). We can take the sum of the constants $c^j_{ik}(0,T)$, $c^j_i(0,T)$, $c^j_0(0,T)$ over all indices as the constant in (20).

□

Lemma 3. *Let the conditions* (5) *hold. Then the following inequalities hold:*

$$\|\int_0^t k_j(\tau)(L_j w)_t(x, t-\tau)\, d\tau\|_{L_p(0,\gamma;L_p(G))} \le c_1 \gamma^{1-1/p} \|k_j\|_{L_p(0,\gamma)} \|w\|_{H(0,\gamma)},\ \gamma \le T, \quad (24)$$

$$\|\int_l^t k_j(\tau)(L_j w)_t(x, t-\tau)\, d\tau\|_{L_p(l,l+\gamma;L_p(G))} \le c_1 \gamma^{1-1/p} \|k_j\|_{L_p(l,l+\gamma)} \|w\|_{H(0,\gamma)},\ l+\gamma \le T, \quad (25)$$

$$\|\int_0^{t-l} k_j(\tau)(L_j w)_t(x, t-\tau)\, d\tau\|_{L_p(l,l+\gamma;L_p(G))} \le c_1 \gamma^{1-1/p} \|k_j\|_{L_p(0,\gamma)} \|w\|_{H(l,l+\gamma)},\ l+\gamma \le T, \quad (26)$$

valid for every every $w \in C([\alpha,\beta]; W_p^2(G))$ *such that* $w_t \in L_p(\alpha,\beta; W_p^2(G))$. *The constant* c_1 *is independent of* γ, l, j.

Proof. To prove (24), we first use the Minkowski inequality inserting the norm in $L_p(G)$ under the integral sign. Next, we apply Lemmas 2 and 3. To establish (25), we use the change of variables. We have

$$\|\int_l^t k_j(\tau)(L_jw)_t(x,t-\tau)\,d\tau\|_{L_p(l,l+\gamma;L_p(G))} = \|\int_0^r k_j(\tau_1+l)(L_jw)_t(x,r-\tau_1)\,d\tau_1\|_{L_p(0,\gamma;L_p(G))}.$$

Thus, we obtain the integral of the form (24) which is estimated similarly. The estimate of the last integral after the change of variables is reduced to estimating the espression

$$\|\int_0^r k_j(\tau)(L_jw)_t(x,r+l-\tau)\,d\tau\|_{L_p(0,\gamma;L_p(G))},$$

which is again of the same form as the left-hand side of (24).

□

Consider the auxiliary equation

$$L_0v_t + L_{0t}v + \int_0^t \sum_{i=1}^m k_i(\tau)(L_iv)_t(x,t-\tau)\,d\tau = f_0. \tag{27}$$

Let $Q^\gamma = G\times(0,\gamma)$. Fix $T_0 \le T$.

Theorem 3. *Assume that $f_0 \in L_p(Q^{T_0})$ $(p>n)$, the conditions (5) hold, and $k_i \in L_p(0,T_0)$ $(i=1,2,\ldots,m)$. Then there exsists a unique solution to the problem (16), (27), such that $v\in H(0,T_0)$. There exists a constant $c>0$ independent of f and T_0 such that a solution to the problem (16), (27) satisfies the estimate*

$$\|v\|_{C([0,T_0];W_p^2(G))} + \|v_t\|_{L_p(0,T_0;W_p^2(G))} \le c\|f_0\|_{L_p(Q^{T_0})}.$$

Proof. We reduce the problem to an integral equation. From (27) we have

$$v(x,t) + L_0^{-1}\int_0^t\int_0^\xi \sum_{i=1}^m k_i(\tau)(L_iv)_\xi(x,\xi-\tau)\,d\tau d\xi = L_0^{-1}\int_0^t f_0(x,\tau)\,d\tau = f_1, \tag{28}$$

where the operator $L_0^{-1}f$ takes a function f onto a solution to the problem (7) with $g=0$. First, we justify a local solvability. We have the equation

$$v + S(v) = f_1 \tag{29}$$

Estimate $\|S(v)\|_{H(0,\gamma)}$ $(\gamma\le T_0)$. Corollary 1 and Lemma 3 yield

$$\|S(v)\|_{H(0,\gamma)} \le c\gamma^{1-1/p}\|v\|_{H(0,\gamma)}\|\vec{k}\|_{L_p(0,\gamma)} \le c\gamma^{1-1/p}\|v\|_{H(0,\gamma)}\|\vec{k}\|_{L_p(0,T_0)}, \tag{30}$$

where the constant c is independent of γ and $\vec{k}=(k_1,k_2,\ldots,k_m)$. Hence, for $\gamma\le\gamma_0$ with $\gamma_0^{1-1/p}c\|\vec{k}\|_{L_p(0,T)} = 1/2$, we obtain that $\|S(v)\|_{H(0,\gamma)} \le \|v\|_{H(0,\gamma)}/2$ and the fixed point theorem implies the solvability of the equation (29) on $[0,\gamma_0]$. Prove that the equation is solvable on every of the segments $[0,l\gamma_0+\tau_0]$, with $l\gamma_0<T_0$, $l=1,2,\ldots$, $\tau_0\le\min(\gamma_0,T_0-l\gamma_0)$. Proceed by induction. Assume that we have already proven the solvability of (29) on the segment $[0,l\gamma_0]$. Rewrite the equation (29) in the form

$$v(x,t) + S_0(v) = f_1 - S(v) + S_0(v) = f_2, \tag{31}$$

$$S_0(v)=\begin{cases} 0, & t\le l\gamma_0,\\ L_0^{-1}\int_{k\gamma_0}^{t}\int_0^{\xi-l\gamma_0}\sum\limits_{i=1}^{m}k_i(\tau)(L_iv)_\xi(x,\xi-\tau)\,d\tau d\xi, & t\in(l\gamma_0,l\gamma_0+\tau_0)\end{cases}. \tag{32}$$

It is easy to make sure that the expression $-S(v)+S_0(v)$ contains the values of the function v on the segment $[0,k\gamma_0]$ only and thereby this expression is an already known function. By Corollary 1 and Lemma 3, the operator $S_0(v)$ admits the estimate

$$\|S_0(v)\|_{H(0,l\gamma_0+\tau_0)}\le c\gamma_0^{1-1/p}\|v\|_{H(l\gamma_0,l\gamma_0+\tau_0)}\|\vec{k}\|_{L_p(0,\gamma_0)}\le\|v\|_{H(0,l\gamma_0+\tau_0)}/2 \tag{33}$$

(as is easily seen, we can assume that the constant c in (33) and that in (30) coincide). Hence, the equation (31) is solvable. Obviously, a solution to the equation (31) with the right-hand side f_2 is an extension of a solution to the equation (29) on the segment $[l\gamma_0,l\gamma_0+\tau_0]$.

□

2. Main Results

Write out additional conditions on the data of the problem. We examine the problem (2) – (4) assuming that the functionals Ψ_j meet the conditions

$$\Psi_j\in L(W_p^2(G),\mathbb{R}),\quad \Psi_j(u_0)=\psi_j(0),\ j=1,2,\ldots,m. \tag{34}$$

The symbol $L(A,B)$ for given spaces A, B stands for the space of linear continuous operators defined on A with values in B.

Well-posedness conditions. Assume that B is the matrix with entries $b_{ij}=\Psi_i(L_0^{-1}L_j(x,0)u_0(x))$ and there exists a constant $\delta_2>0$ such that

$$|\det B|\ge\delta_2\quad\forall t\in[0,T], \tag{35}$$

where $L_0^{-1}f$ is a solution v_0 to the problem $L_0v_0=f,\quad Bv_0|_\Gamma=0$.

Theorem 4. *Assume that $f,f_t\in L_p(Q)$, the conditions (5), (12), (13), (34), (35) hold, $g\in C([0,T];W_p^{s_0}(\Gamma))$, $g_t\in L_p(0,T;W_p^{s_0}(\Gamma))$ $(p>n)$, and $\vec{\psi}\in W_p^1(0,T)$ $(\vec{\psi}=(\psi_1,\psi_2,\ldots,\psi_m)$. Then there exists a unique solution to the problem (2) – (4) such that $u_t\in L_p(0,T;W_p^2(G))$, $u\in C([0,T];W_p^2(G))$, $\vec{k}\in L_p(0,T)$, $\vec{k}=(k_1,k_2,\ldots,k_m)$. For any two solutions $(u_1,\vec{k}_1),(u_2,\vec{k}_2)$ to the problem (2) – (4) relating to the data $f_i,g_i,\vec{\psi}_i$ $(i=1,2)$ satisfying the conditions of the theorem, there is the estimate*

$$\begin{aligned}&\|u_1-u_2\|_{C([0,T];W_p^2(G))}+\|u_{1t}-u_{2t}\|_{L_p(0,T;W_p^2(G))}+\textstyle\sum_{i=1}^m\|k_1-k_2\|_{L_p(0,T)}\le\\ &\le c(\|f_1-f_2\|_{W_p^1(0,T;L_p(G))}+\|g_1-g_2\|_{L_p(0,T;W_p^{s_0}(\Gamma))}+\|\vec{\psi}_1-\vec{\psi}_2\|_{W_p^1(0,T)}),\end{aligned} \tag{36}$$

where the constant c depends, in particular, on the norm of the data in the corresponding spaces and the constants in the condition (35).

Proof. Consider the equivalent problem (see Theorem 2)

$$(L_0v)_t+\int\limits_0^t\sum_{j=1}^m k_i(\tau)((L_jv)_t(x,t-\tau)+(L_j\Phi)_t(x,t-\tau))\,d\tau+\sum_{j=1}^m k_j(t)L_j(x,0)u_0(x)=0, \tag{37}$$

$$Bv|_\Gamma = 0, \quad v(x,0) = 0, \tag{38}$$

$$\Psi_j(v) = \tilde{\psi}_j, \ j = 1, 2, \dots, m. \tag{39}$$

Construct a system for determining the functions k_i. Inverting the operator L_0, we arrive at the equation

$$\begin{gathered} v_t(x,t) + L_0^{-1}L_{0t}v + L_0^{-1}\int_0^t \sum_{j=1}^m k_j(\tau)((L_jv)_t(x,t-\tau) + (L_j\Phi)_t(x,t-\tau))\,d\tau + \\ +v_0(v) = -\sum_{j=1}^m k_j(t)L_0^{-1}L_j(x,0)u_0(x), \end{gathered} \tag{40}$$

where $L_0^{-1}f$ takes a function f onto a solution to the problem (7) with $g = 0$ and the function $v_0(v)$ is zero in the case of the Dirichlet boundary conditions while $v_0(v)$ is a solution to the problem $L_0v_0 = 0$, $Bv_0 = -B_tv$ in the case of the oblique derivative problem (the coefficients of the operator B_t are the derivatives with respect to t of the coefficients of B). The equation (37) can be written in either of the forms

$$\begin{gathered} L_0v(x,t) = -\int_0^t\int_0^\xi \sum_{j=1}^m k_j(\tau)(\tau)((L_jv)_t(x,\xi-\tau) + (L_j\Phi)_t(x,\xi-\tau))\,d\tau d\xi - \\ -\sum_{j=1}^m \int_0^t k_j(\xi)\,d\xi L_j(x,0)u_0(x) = G(v), \end{gathered} \tag{41}$$

$$v(x,t) = L_0^{-1}G(v). \tag{42}$$

Applying the functional Ψ_i to (40), we obtain

$$\begin{gathered} \tilde{\psi}_{it} + \Psi_i(L_0^{-1}L_{0t}v) + \Psi_i(L_0^{-1}\int_0^t \sum_{j=1}^m k_j(\tau)(L_jv)_t(x,t-\tau)\,d\tau) + \Psi_i(v_0) = \\ = -\sum_{j=1}^m k_j(t)\Psi_i(L_0^{-1}L_j(x,0)u_0(x)) - \Psi_i(L_0^{-1}\int_0^t \sum_{j=1}^m k_j(\tau)(L_j\Phi)_t(x,t-\tau)\,d\tau). \end{gathered} \tag{43}$$

In view of (35) this equality implies that

$$\vec{k} = B^{-1}\vec{F} - B^{-1}A(\vec{k}), \tag{44}$$

where $\vec{F}$ has the coordinates $F_i = -\tilde{\psi}_{it} \in W_p^1(0,T)$ $(i = 1, 2, \dots, m)$ and the operator $A(\vec{k})$ the coordinates

$$A_i(\vec{k}) = \Psi_i(L_0^{-1}L_{0t}v) + \Psi_i(L_0^{-1}\int_0^t \sum_{j=1}^m k_j(\tau)((L_jv)_t(x,t-\tau) + (L_j\Phi)_t(x,t-\tau))\,d\tau) + \Psi_i(v_0(v)),$$

where $v = v(\vec{k})$ is a solution to the problem (37), (38). Thus, we have the system for the vector-function $\vec{k} = (k_1, k_2, .., k_m)$. Prove its solvability. Note that by Theorem 3 (used on the segment $[0,\gamma]$), for a given $\vec{k} \in L_p(0,\gamma)$ $(\gamma \le T)$, we can uniquely determine the function $v = v(\vec{k}) \in H(0,\gamma)$ as a solution to the equation (42). Establish some estimates. Let $R = 2\|B^{-1}\vec{F}\|_{L_p(0,T)}$. We look for a vector $\vec{k}$ in the ball $B_R^\gamma = \{\vec{k} \in L_p(0,\gamma) : \|\vec{k}\|_{L_p(0,\gamma)} \leq R\}$. Let $\vec{k} \in B_R^\gamma$. Estimate the quantity $\|L_0^{-1}G(v)\|_{H(0,\gamma)}$. We have

$$G(v) - G(0) = \int_0^t \int_0^\xi \sum_{j=1}^m k_j(\tau)(\tau)(L_j v)_t(x, \xi - \tau)\, d\tau d\xi.$$

Corollary 1, Lemma 4, and the inequalities (11) imply that

$$\|L_0^{-1}(G(v) - G(0))\|_{H(0,\gamma)} = \|L_0^{-1}(G(v) - G(0))\|_{H(0,\gamma)} \leq c_1\gamma^{1-1/p}\|\vec{k}\|_{L_p(0,\gamma)}\|v\|_{H(0,\gamma)}, \quad (45)$$

$$\|L_0^{-1}G(0)\|_{H(0,\gamma)} \leq c_1\gamma^{1-1/p}\|\vec{k}\|_{L_p(0,\gamma)}\|\Phi\|_{H(0,\gamma)} + c_2\|\vec{k}\|_{L_p(0,\gamma)} \ . \quad (46)$$

Here c_1, c_2 are some constants independent of γ and the unknowns. Thus, if

$$\gamma_0^{1-1/p} c_1 R = 1/2, \quad (47)$$

then, for $\gamma \leq \gamma_0$, a solution to the equation (42) satisfies the estimate

$$\|v\|_{H(0,\gamma)} \leq 2\|\vec{k}\|_{L_p(0,\gamma)}(c_1\|\Phi\|_{H(0,T)}\gamma_0^{1-1/p} + c_2) = 2\|\vec{k}\|_{L_p(0,\gamma)}c_3, \quad (48)$$

which can also rewritten as

$$\|v\|_{H(0,\gamma)} \leq 2Rc_3. \quad (49)$$

Proceed with the estimates for the summands in (44). Corollary 1 and the trace theorems [17, Sect. 4.7] yield

$$\|v_0\|_{L_p(\alpha,\beta;W_p^2(G))} \leq c\|B_t v\|_{L_p(\alpha,\beta;W_p^{s_0}(\Gamma))} \leq c_4\|v\|_{L_p(\alpha,\beta;W_p^{2-1/p}(\Gamma))} \leq c_5\|v\|_{L_p(\alpha,\beta;W_p^2(G))}, \quad (50)$$

where the constant c_5 is independent of $0 \leq \alpha < \beta \leq T$. Let $\kappa_0 = \max_i \|\Psi_i\|_{L(W_p^2(G),\mathbb{R})}$. In view of (10), (34), (48), (50), we infer

$$\begin{aligned}\sum_{i=1}^m \|\Psi_i(v_0)\|_{L_p(0,\gamma)} \leq \kappa_0 c_5\|v\|_{L_p(0,\gamma;W_p^2(G))} \leq c_6\gamma^{1-1/p}\|v_t\|_{L_p(0,\gamma;W_p^2(G))} \leq \\ \leq 2c_3c_6\gamma^{1-1/p}\|\vec{k}\|_{L_p(0,\gamma)}.\end{aligned} \quad (51)$$

Using the arguments those in the derivation of (20), (10), Corollary 1, and (48), we obtain

$$\begin{aligned}\sum_{j=1}^m \|\Psi_j(L_0^{-1}L_{0t}v)\|_{L_p(0,\gamma)} \leq \kappa_0 c_7\|v\|_{L_p(0,\gamma;W_p^2(G))} \leq c_8\gamma^{1-1/p}\|v_t\|_{L_p(0,\gamma;W_p^2(G))} \leq \\ \leq 2c_3c_8\gamma^{1-1/p}\|\vec{k}\|_{L_p(0,\gamma)},\end{aligned} \quad (52)$$

where the constant c_8 is independent of γ and $\vec{k} = (k_1, k_2, \ldots, k_m)$. Next, using (34), Corollary 1, and Lemma 3, we have

$$\begin{aligned}\sum_{i=1}^m \|\Psi_i(L_0^{-1}\int_0^t \sum_{j=1}^m k_j(\tau)((L_j v)_t(x, t-\tau) + (L_j\Phi)_t(x, t-\tau))\, d\tau)\|_{L_p(0,\gamma)} \leq \\ \leq \kappa_0 c_1\|\vec{k}\|_{L_p(0,\gamma)}\gamma^{1-1/p}(\|v\|_{H(0,\gamma)} + \|\Phi\|_{H(0,T)}) \leq \|\vec{k}\|_{L_p(0,\gamma)}c_9\gamma^{1-1/p},\end{aligned} \quad (53)$$

where $c_9 = \kappa_0 c_1(2Rc_3 + \|\Phi\|_{H(0,T)})$. The estimates (51) – (53) ensure that

$$\|B^{-1}A(\vec{k})\|_{L_p(0,\gamma)} \leq \|\vec{k}\|_{L_p(0,\gamma)}\gamma^{1-1/p}c_{10}, \ c_{10} = (2c_3(c_6 + c_8) + c_9). \quad (54)$$

Choose $\gamma_1 \leq \gamma_0$ such that $c_{10}\gamma_1^{1-1/p} = 1/2$. In this case, for $\gamma \leq \gamma_1$, the operator $B^{-1}\vec{F} - B^{-1}A(\vec{k})$ takes the ball B_R^γ into itself. Demonstrate that it is also contractive for an appropriate γ. Let $\vec{k}^1, \vec{k}^2 \in B_R^\gamma$ $(\gamma \leq \gamma_1)$. Denote by v_1, v_2 the corresponding solutions to the equation (42). We have

$$L_0 v_i(x,t) = -\int_0^t\int_0^\xi \sum_{j=1}^m k_j^i(\tau)(L_j v_i)_t(x,\xi-\tau)\,d\tau d\xi - \\ -\sum_{j=1}^m \int_0^t k_j^i(\xi)\,d\xi L_j(x,0)u_0(x) - \int_0^t\int_0^\xi \sum_{j=1}^m k_j^i(\tau)(L_j\Phi)_t(x,\xi-\tau)\,d\tau = G(v_i),\ i=1,2. \tag{55}$$

Subtracting these equalities, we obtain that

$$L_0(v_1-v_2) = -\int_0^t\int_0^\xi \sum_{j=1}^m (k_j^1-k_j^2)(\tau)(L_jv_1)_t(x,\xi-\tau) + k_j^2(\tau)(L_jv_1-L_jv_2)_t(x,\xi-\tau)\,d\tau d\xi - \\ -\sum_{j=1}^m\int_0^t (k_j^1-k_j^2)(\xi)\,d\xi L_j(x,0)u_0(x) - \int_0^t\int_0^\xi\sum_{j=1}^m (k_j^1-k_j^2)(\tau)(L_j\Phi)_t(x,\xi-\tau)\,d\tau. \tag{56}$$

As before in (48) (see also (45) and (47)), we have

$$\|v_1-v_2\|_{H(0,\gamma)} \leq 4\gamma^{1-1/p}c_1c_3R\|\vec{k_1}-\vec{k_2}\|_{L_p(0,\gamma)} + 2c_3\|\vec{k_1}-\vec{k_2}\|_{L_p(0,\gamma)} \leq \|\vec{k_1}-\vec{k_2}\|_{L_p(0,\gamma)}4c_3. \tag{57}$$

Thus, we obtain one more additional summand as compared with the estimate (48). Next, consider the difference $A(\vec{k}_1) - A(\vec{k}_2)$. We have

$$A_i(\vec{k}_1) - A_i(\vec{k}_2) = \Psi_i(L_0^{-1}\int_0^t \sum_{j=1}^m k_j^1(\tau)(L_jv_1)_t(x,t-\tau) - k_j^2(\tau)(L_jv_2)_t(x,t-\tau)\,d\tau) + \\ +\Psi_i(L_0^{-1}L_t(v_1-v_2)) + \Psi_i(v_0(v_1-v_2)) + \Psi_i(L_0^{-1}\int_0^t\sum_{j=1}^m (k_j^1-k_j^2)(\tau)(L_j\Phi)_t(x,t-\tau)\,d\tau).$$

Repeating the arguments with use of the estimates (49), (57) (valid for the functions v_1, v_2), we conclude that

$$\|B^{-1}A(\vec{k}^1) - B^{-1}A(\vec{k}^2)\|_{L_p(0,\gamma)} \leq \|\vec{k}^1 - \vec{k}^2\|_{L_p(0,\gamma)}\gamma^{1-1/p}c_{11},\ c_{11} = (c_{10} + 4\kappa_0 c_1 R c_3). \tag{58}$$

Next, we can find $\gamma_2 \leq \gamma_1$ such that $\gamma_2^{1-1/p}c_{11} = 1/2$. In this case, for $\gamma \leq \gamma_2$, the equation (44) is uniquely solvable in the ball B_R^γ.

Proceed with the question of global (in time) solvability of (44). We argue as in the proof of Theorem 3. Demonstrate that there exists $\gamma_3 \leq \gamma_2$ such that the solvability of the system (44) on the segment $[0, l\gamma_3]$ $(l = 1, 2, \ldots,)$ implies the solvability of this system on $[l\gamma_3, l\gamma_3 + \tau_0]$, where $\tau_0 = \min(\gamma_3, T - l\gamma_3)$. Assume that the system is solvable on $[0, l\gamma_3]$ $(l\gamma_3 < T)$. Put

$$\tilde{k}(t) = \begin{cases} \vec{k}(t), & t \in [l\gamma_3, l\gamma_3 + \tau_0] \\ 0, & t < l\gamma_3 \end{cases},\quad \hat{k}(t) = \begin{cases} 0, & t \in [l\gamma_3, l\gamma_3 + \tau_0] \\ \vec{k}(t), & t < l\gamma_3 \end{cases}.$$

The function $\hat{k}(t)$ is already known and we need to determine the function $\tilde{k}(t)$. Using the notation $S_0(v)$ of Theorem 3, where γ_0 is replaced with γ_3, we rewrite (41) in the form

$$v + S_0 v = S_1(\tilde{k}) + f_3, \tag{59}$$

where the right-hand side coincide with $L_0^{-1}G(v)$ for $t \le l\gamma_3$ and, for $t > l\gamma_3$, we have

$$S_1(\tilde{k}) = -L_0^{-1}\int_0^t\int_0^\xi\sum_{j=1}^m \tilde{k}_j(\tau)L_j(v+\Phi)_\xi(x,\xi-\tau)\,d\tau d\xi - \sum_{j=1}^m L_0^{-1}\int_0^t \tilde{k}_j(\xi)\,d\xi L_j(x,0)u_0(x),$$

$$f_3 = -L_0^{-1}\int_0^t\int_0^\xi\sum_{j=1}^m \hat{k}_j(\tau)L_j(v+\Phi)_\xi(x,\xi-\tau)\,d\tau d\xi - \\ -L_0^{-1}\sum_{j=1}^m\int_0^t \hat{k}_j(\xi)\,d\xi L_j(x,0)u_0(x) + S_0(v).$$

Note that the function f_3 is calculated with the use of the values of the functions v, k on the segment $[0, l\gamma_2]$ and thus we can assume that it is a known function. We can see from (59) that v can be expressed trough $\tilde{k}$ linearly. The proof of the estimate (45), Lemma 4, and the definition of the quantity γ_2 imply that

$$\|S_0 v\|_{H(0,l\gamma_3+\tau_0)} \le c_1\gamma_3^{1-1/p}\|\vec{k}\|_{L_p(0,\gamma_3)}\|v\|_{H(l\gamma_3,l\gamma_3+\tau_0)} \le \|v\|_{H(l\gamma_3,l\gamma_3+\tau_0)}/2. \qquad (60)$$

Let $v_1 = (I+S_0)^{-1}S_1(\tilde{k})$. The estimate (60) yields $\|(I+S_0)^{-1}v\|_{H(0,l\gamma_3+\tau_0)} \le 2\|v\|_{H(0,l\gamma_3+\tau_0)}$ for all $v \in H(0, l\gamma_3+\tau_0)$. In this case Corollary 1, Lemma 3, and (47) – (49) imply that

$$\|v_1\|_{H(0,l\gamma_3+\tau_0)} \le 2c_1\gamma_3^{1-1/p}\|\tilde{k}\|_{L_p(l\gamma_3,l\gamma_3+\tau_0)}(\|v\|_{H(0,\gamma_3)} + \|\Phi\|_{H(0,\gamma_3)}) + \\ +2c_2\|\tilde{k}\|_{L_p(l\gamma_3,l\gamma_3+\tau_0)} \le 4c_3\|\tilde{k}\|_{L_p(l\gamma_3,l\gamma_3+\tau_0)}. \qquad (61)$$

Hence, the function $v = v_1 + (I+S_0)^{-1}f_3$ is estimated by the quantity

$$\|v\|_{H(0,l\gamma_3+\tau_0)} \le 4c_3\|\tilde{k}\|_{L_p(l\gamma_3,l\gamma_3+\tau_0)} + 2\|f_3\|_{H(0,l\gamma_3+\tau_0)}. \qquad (62)$$

Write out the representation for $A_j(\vec{k})$ for $t \ge l\gamma_3$. We have

$$A_i(\vec{k}) = A_i(\tilde{k}) + f_{4i},\ A_i(\tilde{k}) = \Psi_i(L_0^{-1}L_{0t}v_1 + v_0(v_1)) + \\ +\Psi_i(L_0^{-1}\int_0^t\sum_{j=1}^m \tilde{k}_j(\tau)L_j(v+\Phi)_t(x,t-\tau)\,d\tau) + \Psi_i(L_0^{-1}\int_0^{t-l\gamma_3}\sum_{j=1}^m \hat{k}_j(\tau)(L_jv_1)_t(x,t-\tau)\,d\tau),$$

$$f_{4i} = \Psi_i(L_0^{-1}L_{0t}(I+S_0)^{-1}f_3) + \Psi_i(v_0((I+S_0)^{-1}f_3)) + \\ +\Psi_i(L_0^{-1}\int_0^t\sum_{j=1}^m \hat{k}_j(\tau)(L_j\Phi)_t(x,t-\tau)\,d\tau) + \Psi_i(L_0^{-1}\int_{t-\gamma_3 l}^t\sum_{j=1}^m \hat{k}_j(\tau)(L_jv)_t(x,t-\tau)\,d\tau) + \\ \Psi_i(L_0^{-1}\int_0^{t-l\gamma_3}\sum_{j=1}^m \hat{k}_j(\tau)(L_j(I+S_0)^{-1}f_3)_t(x,t-\tau)\,d\tau).$$

The function f_{4i} depends on the values of k on the segment $(0, l\gamma_3)$. The remaining expression is a linear operator of the argument $\tilde{k}$. Establish some estimates. Note that the support of v_1 belongs to the segment $[l\gamma_2, l\gamma_2+\tau_0]$. As before on the proof of the estimates (51) and (52), we derive that

$$\sum_{i=1}^m(\|\Psi_i(L_0^{-1}L_{0t}v_1\|_{L_p(l\gamma_3,l\gamma_3+\tau_0)} + \|\Psi_i(v_0(v_1))\|_{L_p(l\gamma_3,l\gamma_3+\tau_0)}) \le \\ \le (c_6+c_8)\gamma_3^{1-1/p}\|v_1\|_{H(l\gamma_3,l\gamma_3+\tau_0)} \le \gamma_3^{1-1/p}(c_6+c_8)4c_3\|\tilde{k}\|_{L_p(l\gamma_3,l\gamma_3+\tau_0)}. \qquad (63)$$

Involving the arguments of the proof of (53), we infer

$$\sum_{i=1}^{m} \|\Psi_i(L_0^{-1}\int_0^t \sum_{j=1}^m \tilde{k}_j(\tau)((L_j v)_t(x,t-\tau)+(L_j\Phi)_t(x,t-\tau))\,d\tau)\|_{L_p(l\gamma_3,l\gamma_3+\tau_0)} \leq$$
$$\leq \kappa_0 c_1 \gamma_3^{1-1/p}\|\tilde{k}\|_{L_p(l\gamma_3,l\gamma_3+\tau_0)}(\|v\|_{H(0,\gamma_3)}+\|\Phi\|_{H(0,T)}) \leq \gamma_3^{1-1/p}\|\tilde{k}\|_{L_p(l\gamma_3,l\gamma_3+\tau_0)}c_9. \tag{64}$$

Moreover, we have

$$\sum_{i=1}^{m} \|\Psi_i(L_0^{-1}\int_0^{t-l\gamma_3} \sum_{j=1}^m \hat{k}_j(\tau)((L_j v_1)_t(x,t-\tau)\,d\tau)\|_{L_p(l\gamma_3,l\gamma_3+\tau_0)} \leq$$
$$\leq \kappa_0 c_1 \gamma_3^{1-1/p}\|\vec{k}\|_{L_p(0,\gamma_3)}4c_3\|\tilde{k}\|_{L_p(l\gamma_3,l\gamma_3+\tau_0)} \leq \kappa_0 c_1 \gamma_3^{1-1/p}2Rc_3\|\tilde{k}\|_{L_p(l\gamma_3,l\gamma_3+\tau_0)}. \tag{65}$$

Thus, the estimate

$$\|B^{-1}A(\tilde{k})\|_{L_p(l\gamma_3,l\gamma_3+\tau_0)} \leq \gamma_3^{1-1/p}\|\tilde{k}\|_{L_p(l\gamma_3,l\gamma_3+\tau_0)}((c_6+c_8)4c_3+c_9+\kappa_0 c_1 2Rc_3)$$

is valid. Hence, if we choose γ_3 so that

$$\gamma_3^{1-1/p}((c_6+c_8)4c_3+c_9+\kappa_0 c_1 2Rc_3)=1/2,$$

then the operator $B^{-1}A(\tilde{k})$ is contractive. Therefore, if the system (44) is solvable on the segment $[0,l\gamma_3]$ ($l=1,2,\ldots,$) then the system is solvable on $[l\gamma_3,l\gamma_3+\tau_0]$, where $\tau_0=\min(\gamma_3,T-l\gamma_3)$. The latter implies that the system (44) is solvable on $[0,T]$. Show that the corresponding function $v=v(k)$ (a solution to the problem (37), (38)) meets the conditions (39). Inverting the operator L_0, we validate the equality (40). Applying the functionals Ψ_j to (40), we obtain that

$$(\Psi_i(v))_t+\Psi_i(L_0^{-1}L_{0t}v)+\Psi_i(L_0^{-1}\int_0^t\sum_{j=1}^m k_j(\tau)(L_j v)_t(x,t-\tau)\,d\tau)+\Psi_i(v_0)=$$
$$=-\sum_{j=1}^m k_j(t)\Psi_i(L_0^{-1}L_j(x,0)u_0(x))-\Psi_i(L_0^{-1}\int_0^t\sum_{j=1}^m k_j(\tau)(L_j\Phi)_t(x,t-\tau)\,d\tau).$$

Subtracting this equality from (43), we arrive at the equality $(\Psi_i(v))_t-\tilde{\psi}_{it}=0$, or (in view of (13), (17)) $\Psi_i(v)=\tilde{\psi}_i$, i. e., the equalities (39) are fulfilled. The claim follows from Theorem 2.

The proof of the estimate (36) is in line with the proof of the existence. We just repeat the arguments paying attention to constants in the corresponding estimates.

□

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ОБРАТНЫЕ ЗАДАЧИ ДЛЯ МАТЕМАТИЧЕСКИХ МОДЕЛЕЙ КВАЗИСТАЦИОНАРНЫХ ЭЛЕКТРОМАГНИТНЫХ ВОЛН В АНИЗОТРОПНЫХ НЕМЕТАЛЛИЧЕСКИХ СРЕДАХ С ДИСПЕРСИЕЙ

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В работе рассматриваются обратные задачи эволюционного типа для математических моделей квазистационарных электромагнитных волн. В модели предполагается, что длина волны мала по сравнению с пространственными неоднородностями. Вводя электрический и магнитный потенциал получаем эллиптическое уравнение второго порядка по пространственным переменным, содержащее интегральные слагаемые типа свертки по времени. После дифференцирования по времени задача сводится к уравнению составного типа с интегральным слагаемым. Определению вместе с решением подлежат неизвестные коэффициенты в интегральном операторе. Дополнительно к краевым условиям задаются условия переопределения в виде заданного набора функционалов от решения, которые могут иметь произвольный вид (интегралы от решения с весом, значения решения в отдельных точках и пр.). В качестве основных пространств рассматриваются пространства С.Л. Соболева. Доказываются теоремы о существовании и единственности решения поставленной задачи в целом по времени, приводится оценка устойчивости.

Ключевые слова: уравнения соболевского типа; эллиптическое уравнение; уравнения с памятью; обратная задача; краевая задача.

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