

**BOUNDARY INVERSE PROBLEM FOR STAR-SHAPED GRAPH
WITH DIFFERENT DENSITIES STRINGS-EDGES**

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The paper is devoted to the mathematical modelling of star-shaped geometric graphs with n rib-strings of different density and the solution of the boundary inverse spectral problem for Sturm–Liouville differential operators on these graphs. Earlier it was shown that if strings have the same length and densities, then the stiffness coefficients of springs at the ends of graph strings are not uniquely recovered from natural frequencies. They are found up to permutations of their places. We show, that if the strings have different densities, then the stiffness coefficients of springs on the ends of graph strings are uniquely recovered from all natural frequencies. Counterexamples are shown that for the unique recovery of the stiffness coefficients of springs on n dead ends of the graph, it is not enough to use n natural frequencies. Examples are also given showing that it is sufficient to use $n + 1$ natural frequencies for the uniqueness of the recovery of the stiffness coefficients of springs at the n ends of the strings. Those, the uniqueness or non-uniqueness of the restoration of the stiffness coefficients of springs at the ends of the strings depends on whether the string densities are identical or different.

Keywords: natural frequencies; star-shaped graph; inverse problems; strings; densities; boundary conditions.

Introduction

The main results on direct and inverse spectral problems for Sturm–Liouville operators on an interval are presented in [1–11] and other works. Differential operators on graphs (networks, trees) often appear in natural sciences and engineering (see [12–22] and the references therein). Most of the results in this direction are devoted to direct problems of studying properties of the spectrum and the root functions for operators on graphs. Inverse spectral problems, because of their nonlinearity, are more difficult to investigate, and nowadays there exists only a small number of papers in this area. In particular, inverse spectral problems of recovering the coefficients of differential operators on trees (i.e. on graphs without cycles) were solved. Boundary inverse eigenvalue problems geometric graphs were considered in [23–27]. In these papers it is shown that if strings have the same length and densities, then the stiffness coefficients of springs at the ends of strings are not uniquely recovered from natural frequencies. They are found up to permutations of their places. We show, that if the strings have different densities, then the stiffness coefficients of springs on the ends of strings are uniquely recovered from natural frequencies. Those, the uniqueness or non-uniqueness of the restoration of the stiffness coefficients of springs at the ends of the strings depend on whether the string densities are identical or different.

1. Eigenvalues for Sturm–Liouville Differential Operators on the Star Graph

Let $\Gamma = \gamma_1 \times \gamma_2 \times \dots \times \gamma_n$ be a star-shaped graph which consists of n finite rays $\gamma_j = \{x_j \in (0, l_j)\}$, $j = 1, \dots, n$, with the origin of each ray identified with the single vertex of the graph. We consider on Γ the equation

$$Ly_j = -\frac{d^2 y_j(x_j)}{dx_j^2} + q(x_j)y_j(x_j) = \rho_j \lambda y_j(x_j) = \rho_j \mu^2 y_j(x_j), \quad x_j \in \gamma_j, \quad (1)$$

defined for functions y satisfying the natural Kirchhoff boundary conditions on the vertex:

$$y_1(0) = \dots = y_n(0), \quad (2)$$

$$y'_1(0) + \dots + y'_n(0) = 0. \quad (3)$$

We further require the boundary conditions on the end-points $x_j = l_j$:

$$y'_j(l_j) + H_j y_j(l_j) = 0, \quad j = 1, \dots, n. \quad (4)$$

Here $\rho_1, \rho_2, \dots, \rho_n$ are different numbers, λ and μ are spectral parameters, $H_j \in \mathbb{C}$. Each potential $q_j(x) \in L(\gamma_j)$ ($j = 1, \dots, n$).

Let $s_j(x, \mu)$, $c_j(x, \mu)$, $j = 1, \dots, n + 1$, be the solutions of (1) on the γ_j which satisfy the initial conditions $s_j(0, \mu) = 1 - c_j(0, \mu) = 0$, $1 - s'_j(0, \mu) = c'_j(0, \mu) = 0$ ($j = 1, \dots, n$).

The general solution on every interval can be represented as

$$y_j(x, \mu) = \alpha_j(\mu)c_j(x, \mu) + \beta_j(\mu)s_j(x, \mu), \quad j = 1, \dots, n. \quad (5)$$

It has to satisfy the first Kirchhoff condition (2) for the first n components (5):

$$\alpha_1(\mu) = \dots = \alpha_n(\mu). \quad (6)$$

We denote $\alpha(\mu) = \alpha_1(\mu) = \dots = \alpha_n(\mu)$.

Boundary conditions on the ends of finite intervals (4) put the following conditions on $\alpha(\mu), \beta_1(\mu), \dots, \beta_2(\mu)$

$$\alpha(\mu) [c'_j(l_j, \mu) + H_j c_j(l_j, \mu)] + \beta_j(\mu) [s'_j(l_j, \mu) + H_j s_j(l_j, \mu)] = 0, \quad j = 1, \dots, n. \quad (7)$$

We define ($i = 1, \dots, n$)

$$\sigma_j(\mu) = s'_j(l_j, \mu) + H_j s_j(l_j, \mu), \quad (8)$$

$$\kappa_j(\mu) = c'_j(l_j, \mu) + H_j c_j(l_j, \mu). \quad (9)$$

Then the condition (8) can be rewritten as

$$\alpha(\mu)\kappa_j(\mu) + \beta_j(\mu)\sigma_j(\mu) = 0, \quad j = 1, \dots, n. \quad (10)$$

It is important to mention, that $\sigma_j(\mu)$ and $\kappa_j(\mu)$ cannot vanish simultaneously: multiply (8) by $c_j(l_j, \mu)$ and subtract (9) multiplied by $s_j(l_j, \mu)$. As the result

$$\sigma_j(\mu)c_j(l_j, \mu) - \kappa_j(\mu)s_j(l_j, \mu) = s'_j(l_j, \mu)c_j(l_j, \mu) - c'_j(l_j, \mu)s_j(l_j, \mu) = 1. \quad (11)$$

The second Kirchhoff condition (3) requires

$$\beta_1(\mu) + \dots + \beta_n(\mu) = 0. \quad (12)$$

The first series of eigenvalues. Consider the case

$$\sigma_1(\mu) \cdot \sigma_2(\mu) \cdot \dots \cdot \sigma_n(\mu) \neq 0. \quad (13)$$

Then it follows from (7) that

$$\beta_j(\mu) = -\alpha(\mu) \frac{c'_j(l_j, \mu) + H_j c_j(l_j, \mu)}{s'_j(l_j, \mu) + H_j s_j(l_j, \mu)}. \quad (14)$$

Then it follows from (12) that

$$\alpha(\mu) \sum_{j=1}^n \frac{c'_j(l_j, \mu) + H_j c_j(l_j, \mu)}{s'_j(l_j, \mu) + H_j s_j(l_j, \mu)} = 0, \quad (15)$$

where $\alpha(\mu)$ is arbitrary.

If $\alpha(\mu) \equiv 0$, then $y_j \equiv 0$ and we don't have eigenvalues.

So eigenvalues are the roots of the equation

$$\sum_{j=1}^n \frac{c'_j(l_j, \mu) + H_j c_j(l_j, \mu)}{s'_j(l_j, \mu) + H_j s_j(l_j, \mu)} = 0. \quad (16)$$

The second series of eigenvalues. Consider the case

$$\sigma_1(\mu) \cdot \sigma_2(\mu) \cdot \dots \cdot \sigma_n(\mu) = 0. \quad (17)$$

Denote

$$J = \{j : \sigma_j(\mu) = 0\}. \quad (18)$$

Then $\kappa_j(\mu) \neq 0$, for $j \in J$, and, to fulfil the (7)

$$\alpha(\mu) = 0. \quad (19)$$

So the general solution in this case has to be in the form

$$y_j(x, \mu) = \beta_j(\mu) s_j(x, \mu), \quad j = 1, \dots, n. \quad (20)$$

But $\sigma_j(\mu) \neq 0, j \notin J$, and as the result, to satisfy (7), it has to be $\beta_j = 0$, and as the result the general solution in such case is

$$y_j(x, \mu) = \beta_j(\mu) s_j(x, \mu), \quad j \in J, \quad (21)$$

$$y_j(x, \mu) \equiv 0, \quad j \notin J, \quad (22)$$

where $\beta_j(\mu)$ are arbitrary.

If $\beta_j \equiv 0 (j \in J)$, then $y_j \equiv 0$ and we don't have eigenvalues.

So eigenvalues are the roots of the equation

$$s_j(x, \mu) = 0, \quad j \in J. \quad (23)$$

2. Uniqueness of Solution for Boundary Inverse Problem

The boundary inverse problem consists of determining H_j from eigenvalues. We note that the eigenvalues of the second series cannot be used to solve the inverse problem, since the equation for determining the eigenvalues of the second series does not contain H_j . Therefore, we assume that we know the eigenvalues from the first series.

Let L denote the Sturm–Liouville eigenvalue problem (1) – (4).

In that follows, the problem of type L with different coefficients in the equation and with different parameters in the boundary forms is denoted by \tilde{L} . Additionally, if a certain symbol denotes an object in the problem L , the same symbol with a tilde denotes its counterpart in problem \tilde{L} .

Theorem 1. *If the eigenvalues of the first series for problem L are equal to those of problem \tilde{L} counting their algebraic multiplicities, then the coefficients of the boundary conditions of L and \tilde{L} are also equal to each other; i.e., $H_j = \tilde{H}_j$ for $j = 1, 2, \dots, n$.*

Proof. If λ_i are eigenvalues of the first series for problem L , then from (16) it follows that λ_i are the roots of the entire function

$$\Delta(\lambda) = \sum_{k=1}^n (c'_k(l_k, \mu) + H_k c_k(l_k, \mu)) \cdot \prod_{j=1; j \neq k}^n (s'_j(l_j, \mu) + H_j s_j(l_j, \mu)). \quad (24)$$

It can be seen that $\Delta(\lambda)$ is an entire function of order 1. Moreover, according to the assumptions of the theorem, the eigenvalues of L and \tilde{L} listed with their algebraic multiplicities are equal to each other. Therefore, the Hadamard factorization theorem implies that $\Delta(\lambda) \equiv C \tilde{\Delta}(\lambda)$, where C is a nonzero constant. It follows that

$$\begin{aligned} \Delta(\lambda) - C \tilde{\Delta}(\lambda) &\equiv \sum_{k=1}^n (c'_k(l_k, \mu) + H_k c_k(l_k, \mu)) \cdot \prod_{j=1; j \neq k}^n (s'_j(l_j, \mu) + \tilde{H}_j s_j(l_j, \mu)) - \\ &- C \cdot \sum_{k=1}^n (c'_k(l_k, \mu) + \tilde{H}_k c_k(l_k, \mu)) \cdot \prod_{j=1; j \neq k}^n (s'_j(l_j, \mu) + \tilde{H}_j s_j(l_j, \mu)). \end{aligned} \quad (25)$$

We have the asymptotic formulas

$$\begin{aligned} c_j(x_j, \mu) &= \cos(\mu x_j) + \frac{1}{\mu} u_j(x_j) \sin(\mu x_j) + O\left(\frac{1}{\mu^2}\right), \\ s_j(x_j, \mu) &= \frac{1}{\mu} \sin(\mu x_j) - \frac{1}{\mu^2} u_j(x_j) \cos(\mu x_j) + O\left(\frac{1}{\mu^3}\right), \\ c'_j(x_j, \mu) &= -\mu \sin(\mu x_j) + u_j(x_j) \cos(\mu x_j) + O\left(\frac{1}{\mu}\right), \\ s'_j(x_j, \mu) &= \cos(\mu x_j) + \frac{1}{\mu} u_j(x_j) \sin(\mu x_j) + O\left(\frac{1}{\mu^2}\right), \quad j = 1, 2, \dots, n, \end{aligned}$$

where $u_j(x_j) = \frac{1}{2} \int_0^{x_j} q(t_j) dt_j$ for sufficiently large $\mu \in \mathbb{R}$ [3]. Hence it follows that the corresponding products of c_j , s_j and their derivatives are linearly independent. From this and (25) it follows that $H_j = \tilde{H}_j$. Thus, the theorem is proved. □

The theorem shows that the boundary conditions of Problem L can be uniquely recovered from the infinite set of eigenvalues. However, computer calculations show that

enough finite number of eigenvalues is sufficient to restore n parameters H_j of L . In this case, for the reconstruction we need to use not n , but $n + 1$ eigenvalues of problem L .

Let us demonstrate that n eigenvalues are not enough for unique identification of n coefficients H_j .

3. Examples and Counterexamples

Example 1. (Counterexample 1) Let $\Gamma = \gamma_1 \times \gamma_2 \times \gamma_3$ be a star-shaped graph which consists of 3 finite rays $\gamma_j = \{x_j \in (0, 1)\}$, $l_j = 1$, $j = 1, 2, 3$, with the origin of each ray identified with the single vertex of the graph. We consider on Γ equation

$$Ly_j = -\frac{d^2 y_j(x_j)}{dx_j^2} = \rho_j \mu^2 y_j(x_j), \quad x_j \in \gamma_j, \quad (26)$$

defined for functions y satisfying the natural Kirchhoff boundary conditions on the vertex:

$$y_1(0) = y_2(0) = y_3(0), \quad (27)$$

$$y_1'(0) + y_2'(0) + y_3'(0) = 0. \quad (28)$$

We further require the boundary conditions on the end-points $x_j = l_j = 1$:

$$y_j'(1) + H_j y_j(1) = 0, \quad j = 1, \dots, n. \quad (29)$$

Here μ is a spectral parameter, $\rho_1 = 1$, $\rho_2 = 2$, $\rho_3 = 3$, $H_j \in \mathbb{C}$.

The solutions of (26) on γ_j which satisfies initial condition $s_j(0, \mu) = 1 - c_j(0, \mu) = 0$, $1 - s_j'(0, \mu) = c_j'(0, \mu) = 0$, ($j = 1, 2, 3$) are functions $c_j(x_j, \mu) = \cos(\mu x_j)$ and $s_j(x_j, \mu) = \frac{1}{\mu} \sin(\mu x_j)$, $j = 1, 2, 3$.

Let us take the first tree eigenvalues of the first series of eigenvalues for problem (26) – (29) $\mu_1 = 0, 48450$, $\mu_2 = 1, 4165$, $\mu_3 = 2, 2721$. Using these values, from (16) we arrive at the system of three equations:

$$\begin{aligned} &(0, 88491H_1 - 0, 22566)(0, 85069H_2 + 0, 56613)(0, 68327H_3 + 0, 11703) + \\ &+(0, 96133H_1 + 0, 88491)(0, 56613H_2 - 0, 79876)(0, 68327H_3 + 0, 11703) + \\ &+(0, 96133H_1 + 0, 88491)(0, 85069H_2 + 0, 56613)(0, 11703H_3 - 1, 4435) = 0, \\ &(0, 15372H_1 - 1, 3996)(0, 10724H_2 - 0, 95274)(-0, 21055H_3 - 0, 44664) + \\ &+(0, 69759H_1 + 0, 15372)(-0, 95274H_2 - 0, 86063)(-0, 21055H_3 - 0, 44664) + \\ &+(0, 69759H_1 + 0, 15372)(0, 10724H_2 - 0, 95274)(-0, 44664H_3 + 3, 8020) = 0, \\ &(-0, 64523H_1 - 1, 73588)(-0, 21696H_2 - 0, 16736)(0, 074565H_3 + 0, 86121) + \\ &+(0, 33625H_1 - 0, 64523)(-0, 16736H_2 + 4, 4801)(0, 074565H_3 + 0, 86120) + \\ &+(0, 33625H_1 - 0, 64523)(-0, 21696H_2 - 0, 16736)(0, 86120H_3 - 3, 4645) = 0, \end{aligned} \quad (30)$$

which solution will be six sets of (H_1, H_2, H_3) :

$$\begin{aligned} &\{H_1 = 0, 081922, H_2 = 3, 4849, H_3 = 6, 0394\}, \\ &\{H_1 = 0, 31023, H_2 = 1, 0390, H_3 = 31, 632\}, \\ &\{H_1 = 1, 0000, H_2 = 2, 0000, H_3 = 3, 0000\}, \\ &\{H_1 = 1, 2269, H_2 = 0, 90496, H_3 = 5, 56392\}, \\ &\{H_1 = -0, 45154 + 0, 17493i, H_2 = -3, 8954 + 5, 6196i, H_3 = -0, 55241 - 2, 2121i\}, \\ &\{H_1 = -0, 45154 - 0, 17493i, H_2 = -3, 8954 - 5, 6196i, H_3 = -0, 55241 + 2, 2121i\}. \end{aligned} \quad (31)$$

We substitute the fourth eigenvalue $\mu_4 = 3,2689$ of the problem (26) – (29) in (16). As the result, we obtain the equation

$$\begin{aligned} &(-0,99191H_1 + 0,41503)(0,038526H_2 + 0,96776)(-0,038005H_3 - 0,92795) + \\ &+ (-0,038840H_1 - 0,99191)(0,96776H_2 - 1,6467)(-0,038005H_3 - 0,92795) + \\ &+ (-0,038840H_1 - 0,99191)(0,038526H_2 + 0,96776)(-0,92795H_3 + 3,65503) = 0. \end{aligned} \quad (32)$$

Let us form a new system of equations from the first two equations of system (30) and equation (32). The solution to this new system of equations is the other 6 sets of solutions:

$$\begin{aligned} &\{H_1 = 1,4493, H_2 = 1,9584, H_3 = 2,5961\}, \\ &\{H_1 = 0,84897 + 1,2383i, H_2 = 0,81210 - 0,30203i, H_3 = 4,3564 - 1,0462i\}, \\ &\{H_1 = 1,0000, H_2 = 2,0000, H_3 = 3,0000\}, \\ &\{H_1 = -0,44254 + 0,06691i, H_2 = 14,573 + 6,2654i, H_3 = -2,9975 - 1,7914i\}, \\ &\{H_1 = -0,44254 - 0,066907i, H_2 = 14,573 - 6,2654i, H_3 = -2,9975 + 1,7914i\}, \\ &\{H_1 = 0,84897 - 1,2383i, H_2 = 0,81210 + 0,30203i, H_3 = 4,3565 + 1,0462i\}. \end{aligned} \quad (33)$$

Thus, n coefficients H_j of the boundary conditions (4) are not uniquely recovered by n eigenvalues of problem L .

However, calculations on concrete examples show that the coefficients H_j of the boundary conditions (4) can be uniquely determined by using $n + 1$ eigenvalues of problem L .

Example 2. Let us take the first four eigenvalues of the first series of eigenvalues for problem (26) – (29) $\mu_1 = 0,48450$, $\mu_2 = 1,4165$, $\mu_3 = 2,2721$, $\mu_4 = 3,2689$. The solution of the system of four equations (30), (32) is the intersection of sets (31) and (33). This intersection consists of a unique solution of system:

$$\{H_1 = 1,0000, H_2 = 2,0000, H_3 = 3,0000\}. \quad (34)$$

Example 3. (Counterexample 2) Let $\Gamma = \gamma_1 \times \gamma_2 \times \gamma_3$ be a star-shaped graph which consists of 3 finite rays $\gamma_j = \{x_j \in (0, 1)\}$, $j = 1, 2, 3$, with the origin of each ray identified with the single vertex of the graph. We consider on Γ equation

$$Ly_j = -\frac{d^2 y_j(x)}{dx_j^2} + q_j(x) y_j(x) = \rho_j \mu^2 y_j(x), \quad x_j \in \gamma_j, \quad (35)$$

defined for functions y satisfying the natural Kirchhoff boundary conditions on vertex:

$$y_1(0) = y_2(0) = y_3(0), \quad (36)$$

$$y_1'(0) + y_2'(0) + y_3'(0) = 0. \quad (37)$$

We further require the boundary conditions on end-points $x_j = l_j = 1$:

$$y_j'(1) + H_j y_j(1) = 0, \quad j = 1, \dots, n. \quad (38)$$

Here μ is a spectral parameter, $q_1(x) = x$, $q_2(x) = x + 2$, $q_3(x) = x + 3$, $\rho_1 = 1$, $\rho_2 = 2$, $\rho_3 = 3$, $H_j \in \mathbb{C}$.

The solutions of (35) on γ_j which satisfies initial condition $s_j(0, \mu) = 1 - c_j(0, \mu) = 0$, $1 - s'_j(0, \mu) = c'_j(0, \mu) = 0$, ($j = 1, 2, 3$) are functions that are expressed in terms of Airy functions:

$$c_1(x, \mu) = -\frac{\text{Bi}(1, -\mu^2) \cdot \text{Ai}(-\mu^2 + x)}{(\text{Bi}(-\mu^2) \cdot \text{Ai}(1, -\mu^2) - \text{Bi}(1, -\mu^2) \cdot \text{Ai}(-\mu^2))} +$$

$$+\frac{\text{Ai}(1, -\mu^2) \cdot \text{Bi}(-\mu^2 + x)}{(\text{Bi}(-\mu^2) \cdot \text{Ai}(1, -\mu^2) - \text{Bi}(1, -\mu^2) \cdot \text{Ai}(-\mu^2))},$$

$$c_2(x, \mu) = -\frac{\text{Bi}(1, -2\mu^2 + 2) \cdot \text{Ai}(-2\mu^2 + x + 2)}{\text{Bi}(-2\mu^2 + 2) \cdot \text{Ai}(1, -2\mu^2 + 2) - \text{Bi}(1, -2\mu^2 + 2) \cdot \text{Ai}(-2\mu^2 + 2)} +$$

$$+\frac{\text{Ai}(1, -2\mu^2 + 2) \cdot \text{Bi}(-2\mu^2 + x + 2)}{\text{Bi}(-2\mu^2 + 2) \cdot \text{Ai}(1, -2\mu^2 + 2) - \text{Bi}(1, -2\mu^2 + 2) \cdot \text{Ai}(-2\mu^2 + 2)},$$

$$c_3(x, \mu) = -\frac{\text{Bi}(1, -3\mu^2 + 3) \cdot \text{Ai}(-3\mu^2 + x + 3)}{\text{Bi}(-3\mu^2 + 3) \cdot \text{Ai}(1, -3\mu^2 + 3) - \text{Bi}(1, -3\mu^2 + 3) \cdot \text{Ai}(-3\mu^2 + 3)} +$$

$$+\frac{\text{Ai}(1, -3\mu^2 + 3) \cdot \text{Bi}(-3\mu^2 + x + 3)}{\text{Bi}(-3\mu^2 + 3) \cdot \text{Ai}(1, -3\mu^2 + 3) - \text{Bi}(1, -3\mu^2 + 3) \cdot \text{Ai}(-3\mu^2 + 3)},$$

$$s_1(x, \mu) = \frac{\text{Bi}(-\mu^2) \cdot \text{Ai}(-\mu^2 + x)}{\text{Bi}(-\mu^2) \cdot \text{Ai}(1, -\mu^2) - \text{Bi}(1, -\mu^2) \cdot \text{Ai}(-\mu^2)} -$$

$$-\frac{\text{Ai}(-\mu^2) \cdot \text{Bi}(-\mu^2 + x)}{\text{Bi}(-\mu^2) \cdot \text{Ai}(1, -\mu^2) - \text{Bi}(1, -\mu^2) \cdot \text{Ai}(-\mu^2)},$$

$$s_2(x, \mu) = \frac{\text{Bi}(-2\mu^2 + 2) \cdot \text{Ai}(-2\mu^2 + x + 2)}{\text{Bi}(-2\mu^2 + 2) \cdot \text{Ai}(1, -2\mu^2 + 2) - \text{Bi}(1, -2\mu^2 + 2) \cdot \text{Ai}(-2\mu^2 + 2)} -$$

$$-\frac{\text{Ai}(-2\mu^2 + 2) \cdot \text{Bi}(-2\mu^2 + x + 2)}{\text{Bi}(-2\mu^2 + 2) \cdot \text{Ai}(1, -2\mu^2 + 2) - \text{Bi}(1, -2\mu^2 + 2) \cdot \text{Ai}(-2\mu^2 + 2)},$$

$$s_3(x, \mu) = \frac{\text{Bi}(-3\mu^2 + 3) \cdot \text{Ai}(-3\mu^2 + x + 3)}{\text{Bi}(-3\mu^2 + 3) \cdot \text{Ai}(1, -3\mu^2 + 3) - \text{Bi}(1, -3\mu^2 + 3) \cdot \text{Ai}(-3\mu^2 + 3)} -$$

$$-\frac{\text{Ai}(-3\mu^2 + 3) \cdot \text{Bi}(-3\mu^2 + x + 3)}{\text{Bi}(-3\mu^2 + 3) \cdot \text{Ai}(1, -3\mu^2 + 3) - \text{Bi}(1, -3\mu^2 + 3) \cdot \text{Ai}(-3\mu^2 + 3)}.$$

Let us take the first three eigenvalues of the first series of eigenvalues for problem (35) – (38) $\mu_1 = 1, 7745$, $\mu_2 = 2, 2430$, $\mu_3 = 3, 4376$. Using these values, from (16) we arrive at the system of three equations:

$$(-0, 12050H_1 - 1, 6315)(0, 47686H_2 - 0, 31384)(0, 26508H_3 - 0, 72008) +$$

$$+(0, 61336H_1 + 0, 0059937)(-0, 42522H_2 - 1, 8172)(0, 26508H_3 - 0, 72008) +$$

$$+(0, 61336H_1 + 0, 0059937)(0, 47686H_2 - 0, 31384)(-0, 80651H_3 - 1, 5816) = 0,$$

$$(-0, 58101H_1 - 1, 8116)(0, 13876H_2 - 0, 88951)(-0, 076473H_3 - 0, 94666) +$$

$$+(0, 39850H_1 - 0, 47863)(-0, 95980H_2 - 1, 0540)(-0, 076473H_3 - 0, 94666) +$$

$$+(0, 39850H_1 - 0, 47863)(0, 13876H_2 - 0, 88951)(-0, 98500H_3 + 0, 88323) = 0,$$

$$(-0, 99573H_1 + 0, 73943)(-0, 21611H_2 - 0, 11628)(-0, 10434H_3 + 0, 80048) +$$

$$+(-0, 065638H_1 - 0, 95554)(-0, 11388H_2 + 4, 5660)(-0, 10434H_3 + 0, 80048) +$$

$$+(-0, 065638H_1 - 0, 95554)(-0, 21611H_2 - 0, 11628)(0, 81475H_3 + 3, 3335) = 0,$$
(39)

which solution will be six sets of (H_1, H_2, H_3) :

$$\begin{aligned} & \{H_1 = 3, 0000, H_2 = 2, 0000, H_3 = 1, 0000\}, \\ & \{H_1 = 3, 3145, H_2 = 0, 29642, H_3 = 4, 6856\}, \\ & \{H_1 = 0, 80324 + 6, 5631 i, H_2 = 0, 86403 - 1, 4760 i, H_3 = 0, 33506 + 1, 3100 i\}, \\ & \{H_1 = -0, 81785 + 0, 49296 i, H_2 = -0, 42624 + 4, 92103 i, H_3 = -0, 12302 - 4, 2522 i\}, \\ & \{H_1 = -0, 81785 - 0, 49295 i, H_2 = -0, 42624 - 4, 9210 i, H_3 = -0, 12302 + 4, 2522 i\}, \\ & \{H_1 = 0, 80324 - 6, 5631 i, H_2 = 0, 86403 + 1, 4760 i, H_3 = 0, 33506 - 1, 3100 i\}. \end{aligned} \quad (40)$$

We substitute the fourth eigenvalue $\mu_4 = 4, 9887$ of problem (35) – (38) in (16). As the result, we obtain equation

$$\begin{aligned} & (0, 22829H_1 + 4, 8126)(0, 081214H_2 + 0, 82564)(0, 099045H_3 - 0, 54720) + \\ & + (-0, 19737H_1 + 0, 21965)(0, 83356H_2 - 3, 8390)(0, 099045H_3 - 0, 54720) + \\ & + (-0, 19737H_1 + 0, 21965)(0, 081213H_2 + 0, 82564)(-0, 55175H_3 - 7, 0481) = 0. \end{aligned} \quad (41)$$

Let us form the new system of equations from the first two equations of system (39) and equation (41). The solution to this new system of equations is the other 6 sets of solutions:

$$\begin{aligned} & \{H_1 = 3, 0000, H_2 = 2, 0000, H_3 = 1, 0000\}, \\ & \{H_1 = 1, 5805 + 2, 4752 i, H_2 = -0, 22203 + 0, 67498 i, H_3 = 1, 0034 - 3, 6902 i\}, \\ & \{H_1 = -0, 059508 + 0, 67472 i, H_2 = -6, 1889 - 6, 1889 i, H_3 = 0, 93238 - 2, 1799 i\}, \\ & \{H_1 = -0, 46343, H_2 = 54, 238, H_3 = 11, 361\}, \\ & \{H_1 = -0, 059508 - 0, 67473 i, H_2 = -6, 1889 + 0, 090512 i, H_3 = 0, 93238 + 2, 1799 i\}, \\ & \{H_1 = 1, 5804 - 2, 4751 i, H_2 = -6, 1889 + 0, 090512 i, H_3 = 1, 0034 + 3, 6901 i\}. \end{aligned} \quad (42)$$

Thus, both systems do not have a unique solution.

Example 4. Let us take the first four eigenvalues of problem (35) – (38) $\mu_1 = 1, 7745$, $\mu_2 = 2, 2430$, $\mu_3 = 3, 4376$, $\mu_4 = 4, 9887$. The solution of the system of four equations (39), (41) is the intersection of sets (40) and (42). This intersection consists of a unique solution:

$$\{H_1 = 3, 0000, H_2 = 2, 0000, H_3 = 1, 0000\}. \quad (43)$$

We note that in the case of an arbitrary infinitely differentiable functions $q_j(x_j)$, in calculating the characteristic determinant (24), instead of the exact values of the linearly independent solutions $c_j(x, \mu)$ and $s_j(x, \mu)$ of equation (1), the main parts of the Taylor series for these solutions are used with respect to the two variables x and μ . Moreover, the coefficients of the boundary conditions (for the inverse problem) and the eigenvalues (for the direct problem) are found with a small error, and the accuracy of the calculation is sufficiently high [28].

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**ГРАНИЧНАЯ ОБРАТНАЯ ЗАДАЧА
ДЛЯ ЗВЕЗДООБРАЗНОГО ГРАФА
СО СТРУНАМИ-РЕБРАМИ РАЗЛИЧНОЙ ПЛОТНОСТИ**

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Работа посвящена математическому моделированию звездообразных геометрических графов с n струнами-ребрами различной плотности и решению граничной обратной спектральной задачи для дифференциальных операторов Штурма – Лиувилля

на этих графах. Ранее было показано, что, если струны имеют одинаковую длину и плотность, то коэффициенты жесткости пружин на концах струн восстанавливаются по собственным частотам неоднозначно. Они находятся с точностью до перестановок их местами. В настоящей статье показано, что, если струны имеют разную плотность, то коэффициенты жесткости пружин на концах струн восстанавливаются по всем собственным частотам однозначно. Приведены контрпримеры, показывающие, что для однозначного восстановления коэффициентов жесткостей пружин на n тупиковых вершинах графа недостаточно использования n собственных частот. Приводятся также примеры, показывающие, что для однозначного восстановления коэффициентов жесткостей пружин на n концах струн достаточно использовать $n + 1$ собственную частоту. Таким образом, однозначность или неоднозначность восстановления коэффициентов жесткостей пружин на концах струн зависят от того, являются ли плотности струн одинаковыми или различными.

Ключевые слова: собственные частоты; звезднообразный граф; обратные задачи; струны; плотности; краевые условия.

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