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ANALYTICAL STUDY OF THE MATHEMATICAL MODEL OF WAVE PROPAGATION IN SHALLOW WATER BY THE GALERKIN METHOD

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Of concern is an initial-boundary value problem for the modified Boussinesq equation (IMBq equation) is considered. The equation is often used to describe the propagation of waves in shallow water under the condition of mass conservation in the layer and taking into account capillary effects. In addition, it is used in the study of shock waves. The modified Boussinesq equation belongs to the Sobolev type equations. Earlier, using the theory of relatively p-bounded operators, the theorem of existence and uniqueness of the solution to the initial-boundary value problem was proved. In this paper, we will prove that the solution constructed by the Galerkin method using the system orthornormal eigenfunctions of the homogeneous Dirichlet problem for the Laplace operator converges *-weakly to an precise solution. Based on the compactness method and Gronwall's inequality, the existence and uniqueness of solutions to the Cauchy–Dirichlet and the Showalter–Sidorov–Dirichlet problems for the modified Boussinesq equation are proved.

*Keywords: modified Boussinesq equation; Sobolev type equation; initial-boundary value problem; Galerkin method; *-weak convergence.*

Introduction

Let $\Omega \subset \mathbb{R}^n$ be a domain with the boundary $\partial\Omega$ of class C^∞ and $T \in \mathbb{R}_+$. In the cylinder $C = \Omega \times (0, T)$, consider the modified Boussinesq equation

$$(\lambda - \Delta)u_{tt} - \alpha^2 \Delta u + u^3 = 0, \quad (x, t) \in \Omega \times (0, T) \tag{1}$$

with homogeneous Dirichlet boundary condition

$$u(x, t) = 0, \quad (x, t) \in \partial\Omega \times (0, T) \tag{2}$$

and initial Cauchy conditions

$$u(x, 0) = u_0(x), \quad u_t(x, 0) = u_1(x), \quad x \in \Omega, \tag{3}$$

where $\lambda, \alpha \in \mathbb{R}$. The equation has many applications in various fields of natural science. For example, it simulates wave propagation in shallow water, taking into account capillary effects. In this case, the function $u = u(x, t)$ determines the wave height. In monograph [1] a linear mathematical model of wave propagation in shallow water is constructed. A (modified) mathematical model of wave propagation in shallow water in a one-dimensional region was investigated in [2] and a soliton solution of equation (1) was obtained. The existence of a unique global solution to the Cauchy problem for equation (1) was proved

[3] for $\lambda = 1, \alpha = 1$. In [4], a similar solution was obtained for describing the interaction of shock waves.

The following generalized Pochhammer–Cree equation was considered in [2]

$$u_{tt} - u_{ttxx} - (f(u))_{xx} = 0,$$

where $f(u)$ is a rational function of u. This equation is used to describe the propagation of a longitudinal strain wave in an elastic rod. In [1] and [2], a solution in the form of solitary waves for the Pochhammer–Cree equation

$$u_{tt} - u_{ttxx} - u_{xx} - \frac{1}{p}\left((u^p)\right)_{xx} = 0,$$

with $p = 2, 3, 5$ was constructed and was numerically investigated the interaction of two solitary wave solutions. For $f(u) = a_1 u + a_2 u^2 + a_3 u^3$ and $f(u) = a_1 u + a_3 u^3 + a_5 u^5$ in [5], explicit solitary wave solutions of the last equation were obtained using method of reduction to an algebraic equation. The bifurcation behavior of the phase portraits for the corresponding traveling wave equation was also investigated. Under various parametric conditions, all explicit formulas for solutions with a solitary wave and solutions with a kink wave were obtained in [6]. Also in [6], an initial-boundary value problem for the generalized Pochhammer–Cree equation

$$u_{tt} - u_{xx} - u_{xxt} - u_{xxtt} = (f(u))_{xx},$$

where f is a non-decreasing function from $\{f \in C^{k+1}(R) : f(0) = 0\}$ was studied. Under an additional condition on $f(u)$ the authors proved the existence of a global solution.

In all the works listed above, an essential condition is the continuous invertibility of the operator at the highest derivative with respect to t. However, the operator $\lambda - \Delta$ can be degenerate. Equations that are not solvable with respect to the highest time derivative, according to [7] are called Sobolev type equations.

Using the theory of p-bounded operators developed by G.A. Sviridyuk and his disciples [8, 9], it was shown in [10] that in appropriately chosen spaces the problem (1) – (3) can be reduced to the initial value problem

$$u(0) = u_0, \quad \dot{u}(0) = u_1 \tag{4}$$

for an abstract semilinear second-order Sobolev type equation

$$L\ddot{u} - Mu + N(u) = 0, \tag{5}$$

where $\dot{u}, \ddot{u}$ are the first and the second derivatives with respect to t, $L = \lambda - \Delta, M = \alpha^2 \Delta, N(u) = u^3$. Then, using the phase space method, a theorem on the existence of a unique local solution was proved. It was also noted that in the case of monotonicity of the operator N, the phase space would be a simple manifold.

Equation (1) belongs to the class of high-order Sobolev type equations [11, 12]. Sobolev type equations are closely related to algebraic-differential equations [13, 14]. Nowadays, more and more often the theory of Sobolev type equations is transferred from bounded domains in the space $\mathbb{R}^n$ to geometric graphs [15] and to the space of differentiable k-forms on Riemannian manifolds [16]. Many physical phenomena [17–19], as well as technical and economic processes [20] are modelled using Sobolev type equations. This explains the enduring interest in them.

The paper is structured as follows, firstly we introduce some preliminary information, then we investigate the existence of a solution to (1) – (3) using the Galerkin and compactness methods [21]. In the next section, we prove the uniqueness of the solution based on the embedding theorem and Gronwall's inequality. In conclusion, a remark about the Showalter–Sidorov problem and a recommendation for choosing a system of basis functions are made.

1. Preliminary Statements

Definition 1. *Let X be some Banach space, X^* the dual space for X with respect to the duality $\langle \cdot, \cdot \rangle$. The sequence $f_n \in X^*$ is called weakly-* converge to $f \in X^*$, if for any $g \in X$, $\langle f_n, g \rangle \to \langle f, g \rangle$ for $n \to \infty$.*

Generally speaking, *-weak convergence is weaker than ordinary weak convergence, however, if X is a reflexive Banach space, then *-weak and weak convergence are equivalent.

Lemma 1. [21] *Let O be a bounded domain in $\mathbb{R}^n_x \times \mathbb{R}_t$, g_l and g be functions from $L^q(O)$, $1 < q < \infty$ such that*
$\|g_l\|_{L^q(O)} \leq C$, $g_l \to g$ a.e. in $L^q(O)$.
Then $g_l \to g$ weakly in $L^q(O)$.

Lemma 2. (Courant Principle) *Let H be a separable Hilbert space of nonzero dimension and the operator $A : H \to H$ be a linear compact self-adjoint one. Since all eigenvalues of A are real and finite-multiple, they can be numbered non-decreasingly*

$$\lambda_{-1} \leq \lambda_{-2} \leq \cdots \leq \lambda_{-n} \ldots \lambda_n \leq \cdots \leq \lambda_2 \leq \lambda_1.$$

Then, for any $n \geq 1$, the following relations hold:

$$\lambda_n = \inf_{H_{n-1}} \sup_{\substack{x \perp H_{n-1} \\ x \neq 0}} \frac{(Ax, x)}{\|x\|^2},$$

$$\lambda_{-n} = \sup_{H_{n-1}} \inf_{\substack{x \perp H_{n-1} \\ x \neq 0}} \frac{(Ax, x)}{\|x\|^2},$$

where H_{n-1} is an arbitrary $(n-1)$-dimensional subspace in H.

Lemma 3. [21] *If $f \in L^p(0, T; X)$ and $\dot{f} \in L^p(0, T; X)$, $1 \leq p \leq \infty$ (X is a Banach space), then f, after perhaps changing on the set from the interval $(0, T)$ with zero measure, is a continuous mapping from $[0, T]$ to X.*

Lemma 4. (Gronwall's lemma [22]) *Let $g(t) \geq 0$ and $f(t) \geq 0$ for $t \geq t_0$, and also $g, f \in C[t_0, +\infty]$, and for $t \geq t_0, c > 0$ the inequality*

$$g(t) \leq c + \int_{t_0}^{t} f(s) g(s) ds,$$

be satisfied. Then the inequality

$$g(t) \leq c e^{\int\limits_{t_0}^{t} f(s)ds}$$

holds. Moreover, if $c = 0$, then $g(t) = 0$.

Lemma 5. (The Rellich–Kondrashov embedding theorem [23]) *Let $\Omega \subset \mathbb{R}^n$ be a domain with a boundary of the class $C^s, s \geq 1$, $s > l$, $1 < p \leq q < \infty$, $s - \frac{n}{p} \geq l - \frac{n}{q}$. Then*

$$W_p^s(\Omega) \subset W_q^l(\Omega) \text{ completely continuous (compact).}$$

Earlier, problem (4), (5) was studied by the methods of p-bounded operators theory. Let X, Y be Banach spaces, the operator $L \in \mathcal{L}(X; Y)$ (i.e., linear and continuous), and the operator $M \in \mathcal{CL}(X; Y)$ (linear and densely defined). The set

$$\rho^L(M) = \{\mu \in \mathbb{C} : (\mu L - M)^{-1} \in \mathcal{L}(Y; X)\}$$

is called the *resolvent set* of the operator M with respect to the operator L (or, the *L-resolvent set* of the operator M). The set $\mathbb{C} \backslash \rho^L(M) = \sigma^L(M)$ is called the *spectrum* of the operator M with respect to the operator L (or, the *L-spectrum* of the operator M).

Operator functions $(\mu L - M)^{-1}, R_\mu^L = (\mu L - M)^{-1} L, L_\mu^L = L(\mu L - M)^{-1}$ with the domain $\rho^L(M)$ are called, respectively, *resolvent*, *right resolvent*, *left resolvent* of the operator M with respect to the operator L (in short, *L-resolvent, right L-resolvent, left L-resolvent of the operator M*).

An operator M is called *(L, σ)-bounded* if

$$\exists a > 0 \ \forall \mu \in \mathbb{C} : (|\mu| > a) \Rightarrow (\mu \in \rho^L(M)).$$

Let the operator M be (L, σ)-bounded. Then the operators

$$P = \frac{1}{2\pi i} \int\limits_\Gamma R_\lambda^L(M) d\lambda \text{ and } Q = \frac{1}{2\pi i} \int\limits_\Gamma L_\lambda^L(M) d\lambda$$

are projectors in the spaces $\mathfrak{U}$ and $\mathfrak{F}$, respectively. Here $\Gamma = \{\lambda \in \mathbb{C} : |\lambda| = r > a\}$.

Definition 2. *The set $\mathfrak{P}$ is called the phase space of equation (5) if*
1) for any $(u_0, u_1) \in T\mathfrak{P}$ ($T\mathfrak{P}$ is the tangent bundle of $\mathfrak{P}$) there is a unique solution to problem (4), (5);
2) any solution $u = u(t)$ of equation (5) lies in $\mathfrak{P}$ as a trajectory.
Moreover, the notation $(u_0, u_1) \in T\mathfrak{P}$ should be understood as $u_0 \in \mathfrak{P}$ and $u_1 \in T_{u_0}\mathfrak{P}$.

Let $\ker L \neq \{0\}$ and the operator M be $(L, 0)$ bounded, then, by the splitting theorem [9], equation (7) can be reduced to an equivalent system of equations

$$\begin{cases} 0 = (\mathbb{I} - Q)(M + N)(u), \\ \ddot{u}^1 = L_1^{-1} Q(M + N)(u), \end{cases}$$

where $u^1 = Pu$. Then the phase space $\mathfrak{P}$ of equation (5) is the set [10]

$$\mathfrak{P} = \{u \in \mathfrak{U} : (\mathbb{I} - Q)(M + N)(u) = 0\}.$$

Thus, the existence of a unique local solution was proved.

Every time when solving initial-boundary value problem for the Sobolev type equation by the Galerkin method, there arises an algebraic-differential system of the following form

$$A\ddot{x} = F(x), \tag{6}$$

where $x(t) \in \mathbb{R}^m, m \in \mathbb{N}, t \in [0, T]$, rank $A = k, k < m$. Transform system (6) to a first-order system introducing a new variable $y(t) \in \mathbb{R}^{2m}$ and new matrix operators

$$y(t) = \begin{pmatrix} \dot{x}(t) \\ x(t) \end{pmatrix}, \quad \bar{A} = \begin{pmatrix} A & \mathbb{O} \\ \mathbb{O} & \mathbb{I} \end{pmatrix}, \quad \bar{F} = \begin{pmatrix} \mathbb{O} & F(\cdot) \\ \mathbb{I} & \mathbb{O} \end{pmatrix}.$$

Then we get

$$\bar{A}\dot{y} = \bar{F}(y), \tag{7}$$

$$\text{rank}\, \bar{A} = k + m. \tag{8}$$

Split system (7) into two subsystems

$$0 = \bar{F}^0(y), \tag{9}$$

$$\bar{Q}\bar{A}\dot{y} = \bar{F}^1(y), \tag{10}$$

where $\bar{F}^0 = \bar{P}\bar{Q}\bar{F}(y)$, $\bar{F}^1(y) = (\mathbb{I} - \bar{P})\bar{Q}\bar{F}(y)$, the matrix $\bar{Q}$ is obtained from the identity matrix by replacing the top rows with basis vectors of the left kernel (cokernel) of the matrix $\bar{A}$, $\bar{P}$ is a projector onto the left kernel of the matrix $\bar{Q}\bar{A}$. Therefore, the solution to system (7) lies in the set $\mathfrak{M} = \{y \in \mathbb{R}^{2m} : \bar{F}^0(y) = 0\}$.

Let the function $\bar{F} \in C^s, s \geq 1$, then the condition

$$\text{rank}\, (\bar{F}^0)'_{y_0} = l, \tag{11}$$

where $(\bar{F}^0)'_{y_0}$ is the Jacobi matrix of the function $\bar{F}^0$ at the point y_0 has sense. Let there exist $y_0 \in \mathfrak{M}$ such that condition (11) is satisfied in some neighborhood $O(y_0) \cap \mathfrak{M}$. Then $O(y_0) \cap \mathfrak{M}$ is a C^s-manifold and equation (9) can be reduced to the form

$$(\bar{F}^0)'_y \dot{y} = 0, \quad y(0) = y_0. \tag{12}$$

Suppose that

$$\ker \bar{Q}\bar{A} \cap \ker (\bar{F}^0)'_{y_0} = \{0\}, \tag{13}$$

in the neighborhood $O(y_0)$. Then the matrix $\bar{Q}A + (\bar{F}^0)'_{y_0}$ is invertible in this neighborhood and the system (10), (12) is reduced to the form

$$\dot{y} = (\bar{Q}\bar{A} + (\bar{F}^0)'_{y_0})^{-1}\bar{F}^1, \quad y(0) = y_0, \tag{14}$$

with a smooth right hand side.

By virtue of [13, Theorem 1], the following theorem holds:

Theorem 1. *Let system (7) satisfy (8), $\bar{F} \in C^s, s \geq 1$ and let there be $y_0 \in \mathfrak{M}$ such that in some neighborhood $O(y_0) \cap \mathfrak{M}$ condition (11) and (13) are satisfied. Then for some $t_0 > 0$ there is at least one solution $y \in C^s(0, t_0; \mathfrak{M})$ such that $y_0 = y(0)$. The set $O(y_0) \cap \mathfrak{M}$ is a C^s-manifold of dimension $2m - l \geq k$. For $s \geq 2$, the solution is unique.*

2. Existence Theorem

In some special cases of a nonlinear term in equation (1), one can not only answer the question about the existence and uniqueness of a solution, but also find this solution. Let us formulate and prove a theorem that answers the question on how to find a solution to (1) – (3)

Further, we need several function spaces such as $L^4(\Omega)$, $H_0^1(\Omega)$. The operator $L : H^1(\Omega) \to H^{-1}(\Omega)$ is given by formula

$$\langle Lu, v\rangle = \int\limits_\Omega (\nabla u \nabla v + \lambda uv)dx.$$

Denote $B = L^4(\Omega) \cap H_0^1(\Omega)$ and $D = H^1(\Omega) \cap \operatorname{coim} L$ (where $\operatorname{coim} L = H^1(\Omega) \ominus \ker L$).

In addition, define spaces of distributions (functions with values in a Banach space) $L^\infty(0, T; B)$ and $L^\infty(0, T; L^2(\Omega))$. Construct the conjugate spaces using the Dunford–Pettis theorem: $(L^\infty(0, T; B))^* \simeq L^1(0, T; L^{\frac{4}{3}}(\Omega) \cup H^{-1}(\Omega))$ and $(L^\infty(0, T; D))^* \simeq L^1(0, T; D^*)$.

Let λ_k be the eigenvalues of the homogeneous Dirichlet problem (2) for the Laplace operator, numbered nonincreasingly taking into account their multiplicity, and φ_k be the corresponding eigenfunctions. In addition, the linear span of $\{\varphi_1, \varphi_2, \ldots, \varphi_m\}$ for $m \to \infty$ is dense in B and orthonormal (in the sense of the inner product in $L^2(\Omega)$).

Theorem 2. *Let $\lambda \in [\lambda_1, +\infty)$, $u_0 \in B$ and $u_1 \in D$ and $(u_0, u_1) \in T_{u_0}\mathfrak{P}$. Then there exists a solution to problem (1) – (3) $u = u(x, t)$ such that $u \in L^\infty(0, T; B)$ and $\dot u \in L^\infty(0, T; D)$.*

Proof. The solution to problem (1) – (3) will be sought in the form of the Galerkin approximation

$$u^m(t) = \sum_{k=1}^m a_k^m(t)\varphi_k. \tag{15}$$

We need to find the coefficients $a_k^m(t)$ from the system of algebraic-differential equations

$$\langle L\ddot u^m, \varphi_k\rangle - \alpha^2\langle \Delta u^m, \varphi_k\rangle + \langle (u^m)^3, \varphi_k\rangle = 0, \quad 1 \le k \le m. \tag{16}$$

Using the expansion of the initial functions in a series by basis functions, we obtain the initial conditions for the system of algebraic-differential equations (16)

$$a_k^m(0) = \beta_k^m, \quad \dot a_k^m(0) = \gamma_k^m, \quad 1 \le k \le m, \tag{17}$$

where $u_0^m = \sum\limits_{k=1}^m \beta_k^m \varphi_k \to u_0$ in B when $m \to \infty$, and $u_1^m = \sum\limits_{k=1}^m \gamma_k^m \varphi_k \to u_1$ in $L^2(\Omega)$ when $m \to \infty$.

Apply Theorem 1 to problem (16), (17). Suppose $\lambda = \lambda_1$ then λ_1 is an eigenvalue of multiplicity 1. Let's write out the matrices

$$\bar A = \begin{pmatrix} \mathbb{O} & \mathbb{O} \\ \mathbb{O} & \mathbb{I}_{2m-1} \end{pmatrix}, \quad \bar P = \begin{pmatrix} \mathbb{I} & \mathbb{O} \\ \mathbb{O} & \mathbb{O}_{2m-1} \end{pmatrix}, \quad \mathbb{I} - \bar P = \begin{pmatrix} \mathbb{O} & \mathbb{O} \\ \mathbb{O} & \mathbb{I}_{2m-1} \end{pmatrix}, \quad \bar Q = \mathbb{I}_{2m},$$

$$\bar F^0 = \begin{pmatrix} -\alpha^2 a_1^m(t) - \langle (u^m)^3, \varphi_1\rangle \end{pmatrix}, \quad \bar F^1 = \begin{pmatrix} -\alpha^2 a_{l+1}^m(t) - \langle (u^m)^3, \varphi_2\rangle \\ \ldots \\ -\alpha^2 a_m^m(t) - \langle (u^m)^3, \varphi_m\rangle \\ -\alpha^2 \dot a_1^m(t) - \langle (\dot u^m)^3, \varphi_1\rangle \\ \ldots \\ -\alpha^2 \dot a_m^m(t) - \langle (\dot u^m)^3, \varphi_m\rangle \end{pmatrix},$$

moreover, $\mathfrak{M} = T\mathfrak{P} = \{y \in R^{2m} : \bar{F}(y) = 0\}$. Each element of the matrices $\bar{F}^0$, $\bar{F}^1$ is a third degree polynomial in the variables a_k^m, therefore $\bar{F}^0 \in C^\infty$ and $\bar{F}^1 \in C^\infty$. Thus, it is easy to check conditions (11) and (13) in a neighborhood contained in $\mathfrak{M}$. Thus, the conditions of Theorem 1 are satisfied, and hence there is a unique local solution $u^m = u^m(t,x)$, $t \in [0, t^m]$.

Let's get a priori estimates. Multiplying equation (16) by $\dot{a}_k^m(t)\,(1 \le k \le m)$ and summing over k from 1 to m, we get

$$\langle L\ddot{u}^m, \dot{u}^m\rangle - \alpha^2\langle \Delta u^m, \dot{u}^m\rangle + \langle (u^m)^3, \dot{u}^m\rangle = 0. \tag{18}$$

Introduce the norm in the space D $(L^2(\Omega) = \mathrm{coim}L \oplus \ker L)$ $|\dot{u}|^2_{H^1} = \langle L\dot{u}, \dot{u}\rangle$. By the Courant principle, this norm is equivalent to the norm induced by the space $H^1(\Omega)$.

Using the self-adjointness of L, Δ, we obtain $2\langle L\ddot{u}^m, \dot{u}^m\rangle = \frac{d}{dt}\langle L\dot{u}^m, \dot{u}^m\rangle$, $2\langle \Delta u^m, \dot{u}^m\rangle = -\frac{d}{dt}\langle \nabla u^m, \nabla u^m\rangle$, $4\langle (u^m)^3, \dot{u}^m\rangle = \frac{d}{dt}\|u^m\|^4_{L^4(\Omega)}$, and equation (18) takes the form

$$\frac{d}{dt}\left[|\dot{u}^m|^2_{H^1} + \alpha^2\langle \nabla u^m, \nabla u^m\rangle + \frac{1}{2}\|u^m\|^4_{L^4}\right] = 0. \tag{19}$$

Integrate it on the segment $[0,t], t \le t_m$

$$|\dot{u}^m|^2_{H^1} + \alpha^2\|u^m\|^2_{H^1_0} + \frac{1}{2}\|u^m\|^4_{L^4} \le |u^m_1|^2_{H^1} + \alpha^2|u^m_0|^2_{H^1_0} + \frac{1}{2}\|u^m_0\|^4_{L^4}.$$

Since the right-hand side of the equality is bounded, the inequality

$$|\dot{u}^m|^2_{H^1} + \alpha^2\|u^m\|^2_{H^1_0} + \frac{1}{2}\|u^m\|^4_{L^4} \le C \tag{20}$$

take place. The constant C is independent of t_m and therefore (20)holds for all $t \in [0, T]$.

Remark 1. Due to (20) the sequence of functions $\dot{u}^m$ is bounded in the space $L^\infty(0,T;D)$, u^m is bounded in $L^\infty(0,T;B)$.

Since u^m and $\dot{u}^m$ are bounded in the spaces $L^\infty(0,T;B)$ and $L^\infty(0,T;D)$, respectively, which are dual spaces to the separable Banach spaces $L^1(0,T;H^{-1}(\Omega) \cup L^{4/3}(\Omega))$ and $L^1(0,T;D^*)$, one can choose *-weakly convergent subsequences u^{m_l} and $\dot{u}^{m_l}$ such that
$u^{m_l} \to u$ *-weakly in $L^\infty(0,T;B)$,
$\dot{u}^{m_l} \to \dot{u}$ *-weakly in $L^\infty(0,T;L^2(\Omega))$.
Moreover, $\dot{u}^{m_l}$ is understood as a generalized derivative in the space of distributions. Also from the boundedness of $\dot{u}^m$ in the space $L^2(0,T;D)$ and u^m in $L^2(0,T;B)$ (by Remark 1 and the properties of Lebesgue spaces) it follows that u^m is bounded in $H^1(C)$. By Lemma 5, we have $H^1(C) \subset L^2(C)$ (a completely continuous embedding). Therefore, we can assume that

$$u^{m_l} \to u \text{ strongly in } L^2(C) \text{ and almost everywhere.} \tag{21}$$

Since the sequence $(u^{m_l})^3$ is bounded in the space $L^\infty(0,T;L^{4/3}(\Omega))$, it converges to some element z of this space

$$(u^{m_l})^3 \to z \ \text{ *-weakly in } L^\infty(0,T;L^{4/3}(\Omega)) \tag{22}$$

Corollary 1. *Put* $O = C$, $g_l = (u^{m_l})^3$, $g = u^3$, *then by Lemma 1, as well as* (21) *and* (22) $z = u^3$.

Now we can go term by term to the limit in (16), setting $m_l = l$. Let k be fixed and $l > k$, we get

$$\langle L\ddot{u}^l, \varphi_k\rangle + \alpha^2\langle \nabla u^l, \nabla\varphi_k\rangle + \langle (u^l)^3, \varphi_k\rangle = 0. \tag{23}$$

By Remark 1, we have the limit transitions

$$\langle \dot{u}^l, \varphi_k\rangle \to \langle u, \varphi_k\rangle \text{ *-weakly in } L^\infty(0,T);$$

$$\langle \nabla u^l, \nabla\varphi_k\rangle \to \langle \nabla u, \nabla\varphi_k\rangle \text{ *-weakly in } L^\infty(0,T)$$

and therefore

$$\langle \ddot{u}^l, \varphi_k\rangle = \frac{d}{dt}\langle \dot{u}^l, \varphi_k\rangle \to \langle \ddot{u}, \varphi_k\rangle \text{ *-weakly in } L^\infty(0,T),$$

and by Corollary 1

$$\langle (u^l)^3, \varphi_k\rangle \to \langle u^3, \varphi_k\rangle \text{ *-weakly in } L^\infty(0,T).$$

Thus, from (23) we deduce

$$\frac{d^2}{dt^2}\langle Lu, \varphi_k\rangle + \alpha^2\langle \nabla u, \nabla\varphi_k\rangle + \langle u^3, \varphi_k\rangle = 0. \tag{24}$$

In view of the density of the system of functions $\{\varphi_k\}_{k=1}^m$ in the space B for $m \to \infty$, and the arbitrariness of the choice of φ_k, the equality holds for an arbitrary $v \in B$

$$\frac{d^2}{dt^2}\langle Lu, v\rangle + \alpha^2\langle \nabla u, \nabla v\rangle + \langle u^3, v\rangle = 0. \tag{25}$$

Due to the expansion of the initial values into a series $u_l(0) = u_l^0 \to u_0$ in $H^1(\Omega)$ and $u_l(0) \to u(0)$ in B, therefore $u(0) = u_0$.

By Remark 1

$$\langle \ddot{u}^l, \varphi_k\rangle \to \langle \ddot{u}, \varphi_k\rangle \quad \text{ *-weakly in } L^\infty(0,T)$$

and, therefore, taking into account Lemma 3, we obtain

$$\langle \dot{u}^l(0), \varphi_k\rangle \to \langle \dot{u}(t), \varphi_k\rangle|_{t=0} = \langle \dot{u}(0), \varphi_k\rangle.$$

On the other hand, due to the expansion of the initial values into a series

$$\langle \dot{u}^l(0), \varphi_k\rangle \to \langle u_1, \varphi_k\rangle.$$

Thus,

$$\langle \dot{u}(0), \varphi_k\rangle = \langle u_1, \varphi_k\rangle, \quad \forall\ k.$$

Therefore the function $u = u(x,t)$ satisfies the equation and initial conditions, i.e. it is the solution of (1) – (3).

□

3. Uniqueness Theorem

Theorem 3. *Under the conditions of Theorem 1 and Lemma 5, the solution to problem (1) – (3) is unique.*

Proof. Let u and v be two different solutions to problem (1) – (3), denote $w = u - v$. Then equation (1) takes the form

$$(\lambda - \Delta)w_{tt} - \alpha^2 \Delta w = v^3 - u^3, \tag{26}$$

and the Cauchy conditions become homogeneous

$$w(x,0) = 0, \quad w_t(x,0) = 0, \quad w \in \Omega. \tag{27}$$

Similarly to the previous section, equation (26) is reduced to the form (19). However, instead of the standard norms of the spaces H^1 and H_0^1, their equivalent, defined by the rule $|\dot{w}|^2_{H^1} = \langle L\dot{w}, \dot{w}\rangle$, $|w|^2_{H_0^1} = \langle \alpha^{-2}\nabla w, \nabla w\rangle$ is used.

$$\frac{d}{dt}\left[|\dot{w}|^2_{H^1} + |w|^2_{H_0^1}\right] = 2\langle v^3 - u^3, \dot{w}^m\rangle. \tag{28}$$

Obviously

$$2\langle v^3 - u^3, \dot{w}^m\rangle \le 6\int\limits_\Omega \sup(|u|^2, |v|^2)|w||\dot{w}|dx.$$

Using the Holder's inequality, we estimate the right-hand side of the previous inequality

$$\int\limits_\Omega \sup(|u|^2, |v|^2)|w||\dot{w}|dx \le C(\||u|^2\|_{L^4} + \||v|^2\|_{L^4})\|w\|_{L^4}\|\dot{w}\|_{L^2},$$

further, using embedding theorems and the properties of the norm, we obtain

$$C(\||u|^2\|_{L^4} + \||v|^2\|_{L^4})\|w\|_{L^4}|\dot{w}\|_{L^2} \le C(|u|^2_{L^4} + |v|^2_{L^4})|w|_{H_0^1}|\dot{w}|_{H^1} \le$$

$$\le C|w|_{H_0^1}|\dot{w}|_{H^1} \le 2C(|w|^2_{H_0^1} + |\dot{w}|^2_{H^1}).$$

Then (28) leads to the inequality

$$\left[|\dot{w}|^2_{H^1} + |w|^2_{H_0^1}\right] \le 2C\int\limits_0^t (|w|^2_{H_0^1} + |\dot{w}|^2_{H^1})ds,$$

whence, by Lemma 4, we have the equality $|\dot{w}|^2_{H^1} + |w|^2_{H_0^1} = 0$. Therefore $w \equiv 0$ and $u \equiv v$. □

Conclusion

Instead of the Cauchy condition (3) for problem (1) – (3), the Showalter–Sidorov condition

$$L(u(0) - u_0) = 0, \quad L(\dot{u}(0) - u_1) = 0 \tag{29}$$

can be posed. Condition (29) is a natural generalization of the Cauchy conditions for Sobolev type equations [24]. By construction of the conditions (29), the existence and uniqueness theorem has less conditions.

Corollary 2. *Let $\lambda \in [\lambda_1, +\infty)$, $u_0 \in B$ and $u_1 \in D$. Then there is a unique solution to problem (1), (2), (29) $u = u(x,t)$ such that $u \in L^\infty(0,T;B)$ and $\dot{u} \in L^\infty(0,T;D)$.*

The number of terms in (15) should be chosen so that the linear span covers the kernel of the operator L.

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АНАЛИТИЧЕСКОЕ ИССЛЕДОВАНИЕ МАТЕМАТИЧЕСКОЙ МОДЕЛИ РАСПРОСТРАНЕНИЯ ВОЛН НА МЕЛКОЙ ВОДЕ МЕТОДОМ ГАЛЕРКИНА

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Рассматривается начально-краевая задача для модифицированного уравнения Буссинеска (уравнения IMBq). Уравнение часто используется для описания распространения волн на мелкой воде при условии сохранения массы в слое и с учетом капиллярных эффектов. Кроме того, оно используется при исследовании ударных волн. Модифицированное уравнение Буссинеска относится к уравнениям соболевского типа. Ранее, используя теорию относительно p-ограниченных операторов было доказано существование и единственность решения начально-краевой задачи. В данной работе мы докажем, что решение, построенное методом Галеркина по системе ортонормированных собственных функций однородной задачи Дирихле для оператора Лапласа, сходится *-слабо к точному решению. Опираясь на метод компактности и неравенство Гронуолла доказано существование и единственность решений задачи Коши – Дирихле и задачи Шоуолтера – Сидорова – Дирихле для модифицированного уравнения Буссинеска.

*Ключевые слова: модифицированное уравнение Буссинеска; уравнения соболевского типа; начально-краевая задача; метод Галеркина; *-слабая сходимость.*

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