ОБЗОРНЫЕ СТАТЬИ

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OSKOLKOV MODELS AND SOBOLEV-TYPE EQUATIONS

T.G. Sukacheva[1,2]
[1]Novgorod State University, Velikiy Novgorod, Russian Federation
[2]South Ural State University, Chelyabinsk, Russian Federation
E-mail: tamara.sukacheva@novsu.ru

This article is a review of the works carried out by the author together with her students and devoted to the study of various Oskolkov models. Their distinctive feature is the use of the semigroup approach, which is the basis of the phase space method used widely in the theory of Sobolev-type equations. Various models of an incompressible viscoelastic fluid described by the Oskolkov equations are presented. The degenerate problem of magnetohydrodynamics, the problem of thermal convection, and the Taylor problem are considered as examples. The solvability of the corresponding initial-boundary value problems is investigated within the framework of the theory of Sobolev-type equations based on the theory for p-sectorial operators and degenerate semigroups of operators. An existence theorem is proved for a unique solution, which is a quasi-stationary semitrajectory, and a description of the extended phase space is obtained. The foundations of the theory of solvability of Sobolev-type equations were laid by Professor G.A. Sviridyuk. Then this theory, together with various applications, was successfully developed by his followers.

Keywords: Oskolkov systems; Sobolev type equations; phase space; incompressible viscoelastic fluid.

Dedicated to the 70th anniversary of the Teacher
Professor Georgy Anatolyevich Sviridyuk

Introduction

A.P. Oskolkov studied the initial-boundary value problems for equations describing the motion of viscoelastic fluids, "which are capable of relaxation of stresses during deformation or exhibit the phenomenon of delayed development of deformations after stress relief" [17]. Relationship between the stress tensor σ and the strain rate tensor D determines the type of fluid, is called determining or rheological one, and has the form

$$(1+\sum_{l=1}^{L}\lambda_l\frac{\partial^l}{\partial t^l})\sigma = 2\nu(1+\sum_{m=1}^{M}\kappa_m\nu^{-1}\frac{\partial^m}{\partial t^m})D - pE,$$

where $\{\lambda_l\}$, $l = 1,\ldots,L$, are relaxation times, $\{\kappa_m\}$, $m = 1,\ldots,M$, are lag times, p is a pressure of the fluid.

The simplest of them are Maxwell fluids ($L = 1,\ M = 0$), Kelvin–Voigt fluid ($L = 0$, $M = 1$) and Oldroyd fluid ($L = M = 1$). In Maxwell's fluids, after the cessation of motion, the stresses decrease as $\exp(-t{\lambda_1}^{-1})$. In Kelvin–Voigt fluids, when stress is relieved, the strain rate decreases as $\exp(-t{\kappa_1}^{-1})$. In Oldroyd fluids, both an exponential relaxation of stresses and exponential deformation lag are observed.

As a result of substitution of the corresponding rheological ratio in equations of motion for an incompressible fluid

$$u_t + (u \cdot \nabla)u = \nabla\sigma + f, \quad \nabla \cdot u = 0,$$

the work [15] obtains a system, which is a generalization of the famous system of Navier–Stokes equations [12], to which the obtained system is reduced when $L = M = 0$.

A.P. Oskolkov (see [16, 19]) investigated solvability of an initial-boundary value problem in Holder and Sobolev spaces in the cylinder $\Omega \times (0,T)$

$$u(x,0) = u_0(x),\ x \in \Omega, \qquad u(x,t) = 0,\ (x,t) \in \partial\Omega \times (0,T),$$

for the linear equation

$$(\lambda - \nabla^2)u_t = \nu\nabla^2 u - \nabla p + f, \quad \nabla \cdot u = 0.$$

Here ν is a positive parameter, $\lambda > -\lambda_1$ (λ_1 is the smallest eigenvalue of the spectral problem $-\nabla^2 v + \nabla p = \lambda v$, $\nabla \cdot v = 0$, $v = 0$ on $\partial\Omega$).

The works [15, 16] investigate the solvability of an initial-boundary value problem for the "quasi-linear" system of equations

$$(\lambda - \nabla^2)u_t = \nu\nabla^2 u - (u \cdot \nabla)u - \nabla p + f, \quad \nabla \cdot u = 0,$$

in the cylinder $\Omega \times (0,T)$, $\Omega \subset \mathbb{R}^n, n = 2,3,4$, for $\lambda > -\lambda_1$ in the space $L^2(\Omega)$.

Further, A.P. Oskolkov together with his students (see, for example, [18]) establishes the theory of global solvability of initial-boundary value problems for flows of Oldroyd fluids ($n = 2$) and Kelvin–Voigt fluids ($n = 3$) on $[0,\infty)$, on the basis of which the theory of attractors and dynamical systems generated by these initial-boundary value problems arose. A.P. Oskolkov together with his students obtained global a priori estimates on the semiaxis $\mathbb{R}_+$ for solutions to the equations of motion of Oldroyd fluids and Kelvin–Voigt fluids.

In [20], the equation of motion of Kelvin–Voigt fluids was reduced to the operator equation

$$L\dot{u} = Mu + F(u) + f(t),\ t \in \mathbb{R}_+,$$

for which a number of non-local problems were investigated.

Previously, using the phase space method, we investigated the first initial-boundary value problem for the equations obtained by Oskolkov that describe the motion of viscoelastic incompressible Kelvin–Voigt fluids of various orders. In an autonomous case, the foundations of such studies were laid in the articles [36, 37], while the non-autonomous case was established in [45]. These works were fundamental for the construction of the theory of solvability of the Cauchy problem for a semilinear autonomous and non-autonomous Sobolev-type equation. As applications of these theories, initial-boundary value problems for equations obtained by A.P. Oskolkov were investigated. Therefore, we refer to such equations as "Oskolkov equations". Initial-boundary value problems for the equations of magnetohydrodynamics generated by the Oskolkov equations were investigated in [7, 10] in the autonomous case and in [11] in the non-autonomous case.

Note that all these models were studied within the framework of the theory of solvability of the Cauchy problem for Sobolev-type equations. And now the theory of such equations is developed in many directions that is confirmed, for example, by the works [4, 48, 50].

This article is a review of the works, which are devoted to the study of various Oskolkov models and carried out by the author together with her students. A distinctive feature of these works is the use of the semigroup approach, which is a base for the phase space method widely used in the theory of Sobolev-type equations.

Therefore, following [11], first of all, we consider one of the problems of magnetohydrodynamics.

1. Oskolkov Models in Magnetohydrodynamics

1.1. Problem Statement

The system of Oskolkov equations

$$\begin{gathered}(1-\varkappa\nabla^2)v_t = \nu\nabla^2 v - (v\cdot\nabla)v + \sum_{l=1}^{K}\beta_l\nabla^2 w_l - \frac{1}{\rho}\nabla p - 2\Omega\times v + \frac{1}{\rho\mu}(\nabla\times b)\times b + f^1,\\ \nabla\cdot v = 0, \quad \nabla\cdot b = 0, \quad b_t = \delta\nabla^2 b + \nabla\times(v\times b) + f^2.\\ \frac{\partial w_l}{\partial t} = v + \alpha_l w_l,\ \alpha_l\in\mathbb{R}_-,\ \beta_l\in\mathbb{R}_+,\ l=\overline{1,\ K},\end{gathered} \tag{1}$$

simulates the flow of an incompressible viscoelastic Kelvin–Voigt fluid [19] of the non-zero order K in the magnetic field of the Earth. Here the vector functions $v = (v_1(x,t), v_2(x,t),\dots,v_n(x,t))$ and $b = (b_1(x,t), b_2(x,t),\dots,b_n(x,t))$ characterize the fluid velocity and magnetic induction, respectively, $p = p(x,t)$ is the pressure, $\varkappa$ is the coefficient of elasticity, ν is the coefficient of viscosity, Ω is the corner velocity, δ is the magnetic viscosity, μ is the magnetic permeability, ρ is the density, and the parameters β_l, $l = \overline{1,\ K}$, determine the time of pressure retardation (delay). The free terms $f^1 = (f_1^1,\dots,f_n^1), f_i^1 = f_i^1(x,t), f^2 = f^2(x,t)$ respond to external effects on the fluid.

Consider the first initial-boundary value problem for system (1)

$$\begin{aligned} &v(x,0) = v_0(x), \quad b(x,0) = b_0(x), \quad w_l(x,0) = w_{l0}(x),\ x\in D,\\ &v(x,t) = 0, \quad b(x,t) = 0, \quad w_l(x,t) = 0,\ (x,t)\in\partial D\times\mathbb{R}_+,\ l=\overline{1,\ K}\end{aligned} \tag{2}$$

under the assumption that $\mu = 1$ and $\rho = 1$. Here $D\subset\mathbb{R}^n$ is a bounded domain with the boundary ∂D of the class C^∞.

Problems similar to problem (1), (2) arise, for example, in mathematical modelling in geophysical sciences [5].

Note that degenerate models of magnetohydrodynamics were previously studied in the works [6, 8, 32, 34]. A distinctive feature of this work is the presence of the vector-functions $f^1 = (f_1^1,\dots,f_n^1), f_i^1 = f_i^1(x,t), f^2 = f^2(x,t)$ in the right part of equation (1). The model of magnetohydrodynamics taking into account various external influences at $K = 0$ was considered in [9]. The case of $K > 0$ is investigated for the first time in [11].

Problem (1), (2) is investigated in the framework of the theory of semilinear Sobolev-type equations [36, 37]. The main tool of the study is the notion of a relative p-sectorial operator and the resultant resolving degenerate semigroup of operators [42, 46]. The theorem on the existence of a unique solution to this problem is proved and the description of its extended phase space is obtained.

This work continues investigations of magnetohydrodynamics models, which were begun in [6, 32]. The distinctive feature is the existence of the vector-functions $f^k = (f_1^k, f_2^k, \ldots, f_n^k)$ in the right part of equations (1).

1.2. Cauchy Problem for Sobolev-Type Equations

Suppose that $\mathcal{U}$ and $\mathcal{F}$ are Banach spaces, the operator $L \in \mathcal{L}(\mathcal{U};\mathcal{F})$, i.e. L is linear and continuous; the operator $M : \operatorname{dom} M \to \mathcal{F}$ is linear and closed and, moreover, M is densely defined in $\mathcal{U}$, i.e. $M \in \mathcal{C}l(\mathcal{U};\mathcal{F})$. Denote $\mathcal{U}_M = \{u \in \operatorname{dom} M : \|u\| = \|Mu\|_{\mathcal{F}} + \|u\|_{\mathcal{U}}\}$. Let the operator $F \in C^\infty(\mathcal{U}_M;\mathcal{F})$. We suppose that the operator $F \in \mathcal{C}^\infty(\mathcal{U}_M;\mathcal{F})$, and the function $f \in \mathcal{C}^\infty(\bar{\mathbb{R}}_+;\mathcal{F})$.

Consider the Cauchy problem

$$u(0) = u_0 \tag{3}$$

for the semi-linear non-stationary Sobolev-type equation

$$L\dot{u} = Mu + F(u) + f(t). \tag{4}$$

By a *local solution* (hereinafter *solution*) to problem (3), (4) we mean the vector-function $u \in \mathcal{C}^\infty((0,T);\mathcal{U}_M)$ satisfying equation (4) and such that $u(t) \to u_0$ at $t \to 0+$.

We assume that the operator M is strongly (L,p)-sectorial [14]. It is well known that under this condition a solution to problem (3), (4) may be non-unique [41]. Therefore, we seek only solutions to problem (3), (4), which are *quasi-stationary semi-trajectories.*

Definition 1. *Let the space $\mathcal{U}$ be splitted into the direct sum $\mathcal{U} = \mathcal{U}_0 \oplus \mathcal{U}_1$ such that* $\ker L \subset \mathcal{U}_0$. *A solution $u = v + w$ to equation* (4), *where $v(t) \in \mathcal{U}_0$, and $w(t) \in \mathcal{U}_1$ for all $t \in (0,T)$, is called a quasi-stationary semi-trajectory if $L\dot{v} \equiv 0$.*

Also, it is well known [14] that a solution to problem (3), (4) does not exist for all $u_0 \in \mathcal{U}_M$. Therefore, we introduce another definition.

Definition 2. *The set $\mathcal{B}^t \subset \mathcal{U}_M \times \bar{\mathbb{R}}_+$ is called an extended phase space of equation* (4), *if, for any point $u_0 \in \mathcal{U}_M$ such that $(u_0, 0) \in \mathcal{B}^0$, there exists a unique solution to problem* (3), (4), *and $(u(t), t) \in \mathcal{B}^t$.*

We consider problem (3), (4) under the condition that the operator M is strongly (L,p)-sectorial [14]. It is well known that in this case there exists a solution to this problem not for all $u_0 \in \mathcal{U}_M$, and the solution may be non-unique, even if a solution exists. So we introduce two definitions (extended phase space and quasi-stationary semi-trajectory).

It is well known that if the operator M is strongly (L,p)-sectorial, then $\mathcal{U} = \mathcal{U}^0 \oplus \mathcal{U}^1$, $\mathcal{F} = \mathcal{F}^0 \oplus \mathcal{F}^1$, where $\mathcal{U}^0 = \{\varphi \in \mathcal{U} : U^t\varphi = 0 \ \exists t \in \mathbb{R}_+\}$, $\mathcal{F}^0 = \{\psi \in \mathcal{F} : F^t\psi = 0 \ \exists t \in \mathbb{R}_+\}$ are *kernels*, and $\mathcal{U}^1 = \{u \in \mathcal{U} : \lim\limits_{t \to 0+} U^t u = u\}$, $\mathcal{F}^1 = \{f \in \mathcal{F} : \lim\limits_{t \to 0+} F^t f = f\}$ are *images* of the analytic solving semigroups

$$U^t = \frac{1}{2\pi i}\int_\Gamma R^L_\mu(M) e^{\mu t} d\mu, \quad F^t = \frac{1}{2\pi i}\int_\Gamma L^L_\mu(M) e^{\mu t} d\mu$$

(here $\Gamma \subset S^L_{\Theta,a}(M)$ is a contour such that $\arg\mu \to \pm\Theta$ when $|\mu| \to +\infty$) of the linear homogeneous equation

$$L\dot{u} = Mu. \tag{5}$$

Denote by $L_k(M_k)$ the restriction of the operator $L(M)$ on $\mathcal{U}^k$ ($\mathcal{U}^k \cap \operatorname{dom} M$), $k = 0, 1$. Then $L_k : \mathcal{U}^k \to \mathcal{F}^k$, $M_k : \mathcal{U}^k \cap \operatorname{dom} M \to \mathcal{F}^k$, $k = 0, 1$, and the restrictions M_0 and L_1 of the operators M and L on the spaces $\mathcal{U}^0 \cap \operatorname{dom} M$ and $\mathcal{U}^1$, respectively, are linear continuous operators and have bounded inverse operators.

Therefore, problem (3), (4) is reduced to the equivalent system, which we call the *normal form* of problem (3), (4):

$$\begin{aligned} R\dot{u}^0 &= u^0 + G(u) + g(t), \quad u^0(0) = u_0^0, \\ \dot{u}^1 &= Su^1 + H(u) + h(t), \quad u^1(0) = u_0^1, \end{aligned} \tag{6}$$

where $u^k \in \mathcal{U}^k, k = 0, 1$, $u = u^0 + u^1$, the operators $R = M_0^{-1}L_0$, $S = L_1^{-1}M_1$, $G = M_0^{-1}(I - Q)F$, $H = L_1^{-1}QF$, $g = M_0^{-1}(I - Q)f$, $h = L_1^{-1}Qf$.

Here $Q \in \mathcal{L}(F)(\equiv \mathcal{L}(F; F))$ is the projector that splits the space $\mathcal{F}$ as required.

So we study the quasi-stationary semi-trajectories of equation (4) for which $R\dot{u}^0 \equiv 0$. Then we assume that the operator R is *bi-splitting*, i.e. its kernel ker R and image $\operatorname{im} R$ are completed in the space $\mathcal{U}$. Let $\mathcal{U}^{00} = \ker R$, and $\mathcal{U}^{01} = \mathcal{U}^0 \ominus \mathcal{U}^{00}$ be a complement of the subspace $\mathcal{U}^{00}$. Then the first equation of normal form (6) is reduced to

$$R\dot{u}^{01} = u^{00} + u^{01} + G(u) + g(t), \tag{7}$$

where $u = u^{00} + u^{01} + u^1$.

Theorem 1. *Let the operator M be strongly (L, p)-sectorial, and the operator R be bi-splitting. Suppose that there exists a quasi-stationary semi-trajectory $u = u(t)$ of equation (4). Then $u = u(t)$ satisfies the relation*

$$0 = u^{00} + u^{01} + G(u) + g(t), u^{01} = \text{const}. \tag{8}$$

It is known that under the condition of strong (L, p)-sectoriality of the operator M, the operator S is sectorial. Therefore, on $\mathcal{U}^1$, the operator S generates an analytic semigroup, which we denote by $\{U_1^t : t \geq 0\}$, since in reality the operator U_1^t is a restriction of the operator U^t on $\mathcal{U}^1$.

From the fact that $\mathcal{U} = \mathcal{U}^0 \oplus \mathcal{U}^1$ it follows that there exists a projector $P \in \mathcal{L}(\mathcal{U})$ corresponding to this splitting. It is easy to see that $P \in \mathcal{L}(\mathcal{U}_M)$. Then the space $\mathcal{U}_M$ splits into the direct sum $\mathcal{U}_M = \mathcal{U}_M^0 \oplus \mathcal{U}_M^1$ such that the embedding $\mathcal{U}_M^k \subset \mathcal{U}^k$, $k = 0, 1$, is dense and continuous. Further, A'_v denotes the Frechet derivative at the point $v \in \mathcal{V}$ of the operator A defined on the Banach space $\mathcal{V}$.

Theorem 2. *Let the operator M be strongly (L, p)-sectorial, the operator R be bi-splitting, the operator $F \in \mathcal{C}^\infty(\mathcal{U}_M; \mathcal{F})$, and the vector-function $f \in \mathcal{C}^\infty(\mathbb{R}_+; \mathcal{F})$. Suppose that the following conditions hold:*

(i) *In the neighborhood $\mathcal{O}_{u_0} \subset \mathcal{U}_M$ of the point u_0, the following relation takes place:*

$$0 = u_0^{01} + (I - P_R)(G(u^{00} + u_0^{01} + u^1) + g(t)).$$

(ii) *The projector $P_R \in \mathcal{L}(\mathcal{U}_M^0)$, and the operator $I + P_R G'_{u_0^0} : \mathcal{U}_M^{00} \to \mathcal{U}_M^{00}$ is the topological linear isomorphism ($\mathcal{U}_M^{00} = \mathcal{U}_M \cap \mathcal{U}^{00}$).*

(iii) *For the analytic semigroup $\{U_1^t : t \geq 0\}$, the following condition holds:*

$$\int_0^\tau \|U_1^t\|_{\mathcal{L}(\mathcal{U}^1; \mathcal{U}_M^1)} dt < \infty \quad \forall \tau \in \mathbb{R}_+. \tag{9}$$

Then there exists a unique solution to problem (3), (4), *which is a quasi-stationary semi-trajectory.*

Remark 1. For ordinary analytic semigroups having the estimate $\|U_1^t\|_{\mathcal{L}(\mathcal{U}^1;\mathcal{U}^1_M)} < \mathrm{const}/t$, condition (9) is not satisfied. Denote by $\mathcal{U}^1_\alpha = [\mathcal{U}^1;\mathcal{U}^1_M]_\alpha$, $\alpha \in [0,1]$, an interpolation space constructed by the operator S. In Theorem 2, the condition $F \in \mathcal{C}^\infty(\mathcal{U}^1_M;\mathcal{F})$ is completed by condition $H \in \mathcal{C}^\infty(\mathcal{U}^1_M;\mathcal{U}^1_\alpha)$, and the condition (9) is replaced by

$$\int_0^\tau \|U_1^t\|_{\mathcal{L}(\mathcal{U}^1;\mathcal{U}^1_\alpha)}\, dt < \infty,\ \tau \in \mathbb{R}_+. \tag{10}$$

Then the statement of Theorem 2 does not change.

Now let $\mathcal{U}_k$ and $\mathcal{F}_k$ be Banach spaces, the operators $A_k \in \mathcal{L}(\mathcal{U}_k, \mathcal{F}_k)$, and the operators B_k : $\mathrm{dom}\, B_k \to \mathcal{F}$ be linear and closed with domains of definitions $\mathrm{dom}\, B_k$ dense in $\mathcal{U}_k$, $k = 1,2$. Let us construct the spaces $\mathcal{U} = \mathcal{U}_1 \times \mathcal{U}_2$, $\mathcal{F} = \mathcal{F}_1 \times \mathcal{F}_2$ and the operators $L = A_1 \otimes A_2$, $M = B_1 \otimes B_2$. By construction, the operator $L \in \mathcal{L}(\mathcal{U};\, \mathcal{F})$, and the operator $M : \mathrm{dom}\, M \to \mathcal{F}$ is linear, closed and densely defined, $\mathrm{dom}\, M = \mathrm{dom}\, B_1 \times \mathrm{dom}\, B_2$.

Theorem 3. *Let the operators B_k be strongly (A_k, p_k)-sectorial, $k = 1,2$ and $p_1 \geq p_2$. Then the operator M is strongly (L, p_1)-sectorial.*

1.3. Reduction to Abstract Cauchy Problem

In order to reduce problem (1), (2) to problem (3), (4), we move from system (1) to the system

$$\begin{aligned} &(1 - \varkappa\nabla^2)v_t = \nu\nabla^2 v - (v\cdot\nabla)v + \sum_{l=1}^K \beta_l \nabla^2 w_l - \bar{p} - 2\Omega \times v + (\nabla\times b)\times b + f^1,\\ &\nabla(\nabla\cdot v) = 0, \quad \nabla(\nabla\cdot b) = 0, \quad b_t = \delta\nabla^2 b + \nabla\times(v\times b) + f^2,\\ &\frac{\partial w_l}{\partial t} = v + \alpha_l w_l,\ \alpha_l \in \mathbb{R}_-,\ \beta_l \in \mathbb{R}_+,\ l = \overline{1,\ K}. \end{aligned} \tag{11}$$

Now we are interested in solvability of (11), (2). Following the works [14, 43], we introduce the spaces $\mathbf{H}^2_\sigma$, $\mathbf{H}^2_\pi$, $\mathbf{H}_\sigma$ and $\mathbf{H}_\pi$. Here $\mathbf{H}^2_\sigma$ and $\mathbf{H}_\sigma$ are subspaces of solenoid functions in the spaces $(W_2^2(D))^n \cap$ $\cap\, (\overset{\circ}{W}{}_2^1(D))^n$ and $(L_2(D))^n$, respectively, and $\mathbf{H}^2_\pi$ and $\mathbf{H}_\pi$ are their orthogonal (in sense of $(L_2(D))^n$) complements. Denote by Σ the orthoprojector on $\mathbf{H}_\sigma$, and denote its restriction on the space $(W_2^2(D))^n \cap (\overset{\circ}{W}{}_2^1(D))^n$ by Σ also. Let $\Pi = \mathbb{I} - \Sigma$. By the equality $A = \nabla^2 E_n : \mathbf{H}^2_\sigma \oplus \mathbf{H}^2_\pi \to \mathbf{H}_\sigma \oplus \mathbf{H}_\pi$, where E_n is the unit matrix of the order n, we define a linear continuous matrix operator with the discrete finite-time spectrum $\sigma(A) \subset \mathbb{R}$ condensing only on $-\infty$. The formula $B_v : v \to \nabla(\nabla\cdot v)(B_b :\ b \to \nabla(\nabla\cdot b))$ gives a linear continuous surjective operator $B_v(B_b) : \mathbf{H}^2_\sigma \oplus \mathbf{H}^2_\pi \to \mathbf{H}_\pi$ with the kernel $\ker B_v = B_b = \mathbf{H}^2_\sigma$. Using the natural isomorphism of the direct sum and the Cartesian product of Banach spaces, we introduce the spaces $\mathcal{U}_{10} = \mathbf{H}^2_\sigma \times \mathbf{H}^2_\pi \times \mathbf{H}_p$, $\mathcal{F}_{10} = \mathbf{H}_\sigma \times \mathbf{H}_\pi \times \mathbf{H}_p$, where $\mathbf{H}_p = \mathbf{H}_\pi$; $\mathcal{U}_{1i} = \mathbf{H}^2 \cap \overset{\circ}{\mathbf{H}}{}^1 = \mathbf{H}^2_\sigma \times \mathbf{H}^2_\pi$, и $\mathcal{F}_{1i} = \mathbf{L}_2 = \mathbf{H}_\sigma \times \mathbf{H}_\pi$, $i = \overline{1,\ K}$. Then we arrive at the spaces $\mathcal{U}_1 = \oplus_{l=0}^K \mathcal{U}_{1l}$, $\mathcal{F}_1 = \oplus_{l=0}^K \mathcal{F}_{1l}$.

The operators A_1 and B_1 are defined by the formulas $A_1 = \mathrm{diag}\,[\hat{A}_1, E_k]$, where

$$\hat{A}_1 = \begin{pmatrix} \check{A}_1 & O \\ O & O \end{pmatrix}, \quad \check{A}_1 = \begin{pmatrix} \Sigma(\mathbb{I}-\lambda A)\Sigma & \Sigma A(\mathbb{I}-\lambda A)\Pi \\ \Pi(\mathbb{I}-\lambda A)\Sigma & \Pi A(\mathbb{I}-\lambda A)\Pi \end{pmatrix};$$

$B_1 = (B_1^{ij})^2_{i,j=1}$, where

$$B_1^{11} = \begin{pmatrix} \nu\Sigma A & \nu\Sigma A & O \\ \nu\Pi A & \nu\Pi A & -\mathbb{I} \\ O & B & O \end{pmatrix}, \quad B_1^{12} = \begin{pmatrix} \beta_1\Sigma A & \dots & \beta_K\Sigma A \\ \beta_1\Pi A & \dots & \beta_K\Pi A \\ O & \dots & O \end{pmatrix}.$$

In the matrix B_1^{11}, $B = \nabla(\nabla\cdot v) - \nabla(\nabla\cdot b) = B_v - B_b$. The matrix B_1^{21} contains K rows of the form $(\mathbb{I},\ \mathbb{I},\ O)$, $B_1^{22} = \mathrm{diag}\,[\alpha_1,\dots,\alpha_K]$.

Remark 2. Denote by A_σ the restriction of ΣA on $\mathbf{H}^2_\sigma$. According to Solonnikov–Vorovich–Yudovich theorem, the spectrum $\sigma(A_\sigma)$ is real, discrete, finite-time, and condenses only on $-\infty$.

Theorem 4. *i) The operators A_1, B_1 belong to $\mathcal{L}(\mathcal{U}_1;\mathcal{F}_1)$ and if $\varkappa^{-1}\notin\sigma(A)$, then the operator A_1 is bi-splitting,* $\ker A_1 = \{0\}\times\{0\}\times\mathbf{H}_p\times\underbrace{\{0\}\times\ldots\times\{0\}}_{K}$, $\mathrm{im}\,A_1 = \mathbf{H}_\sigma\times\mathbf{H}_\pi\times\{0\}\times\mathcal{F}_1\times\mathcal{F}_2\times\ldots\times\mathcal{F}_K$.

ii) If $\lambda^{-1}\notin\sigma(A)\cup\sigma(A_\sigma)$, then the operator B_1 is $(A_1,1)$-bounded.

Remark 3. The concept of a (L,p)-bounded operator can be found, for example, in [14].

Next, we set $\mathcal{U}_2 = \mathcal{F}_2 = L_2(D)$ and define the linear closed and densely defined operator B_2 by the equality $B_2 = \delta\nabla^2 : \mathrm{def}\,B_2 \to \mathcal{F}_2$, $\mathrm{dom}\,B_2 = W_2^2(D)\cap \overset{\circ}{W}{}_2^1(D)$. Let $A_2 \equiv \mathbb{I}$.

Theorem 5. *The operator B_2 is strongly A_2-sectorial.*

Let $\mathcal{U} = \mathcal{U}_1\times\mathcal{U}_2$, $\mathcal{F} = \mathcal{F}_1\times\mathcal{F}_2$. The vector u of the space $\mathcal{U}$ has the form $u = \mathrm{col}(u_\sigma,u_\pi,u_p,w_1,\dots,w_K,u_b)$, where $\mathrm{col}(u_\sigma,u_\pi,u_p,w_1,\dots,w_K)\in\mathcal{U}_1$, and $u_b\in\mathcal{U}_2$, $u_b = (b_\sigma,b_\pi)$, $b_\sigma\in\mathbf{H}^2_\sigma$, $b_\pi\in\mathbf{H}^2_\pi$. The vector $f\in\mathcal{F}$ has a similar form. The operators L and M are defined by the equalities $L = A_1\otimes A_2$ and $M = B_1\otimes B_2$. The operator L belongs to $\mathcal{L}(\mathcal{U};\mathcal{F})$, and the operator $M:\ \mathrm{dom}\,M\to\mathcal{F}$ is linear, closed and densely defined, $\mathrm{dom}\,M = \mathcal{U}_1\times\mathrm{dom}\,B_2$.

Theorem 6. *Let $\varkappa^{-1}\notin\sigma(A)$, then the operator M is strongly $(L,1)$-sectorial.*

We proceed to the construction of a nonlinear operator F. Represent

$$F = F_1\otimes F_2,$$

where

$$F_1 = F_1(u_\sigma,u_\pi,b) = col(-\Sigma(((u_\sigma+u_\pi)\cdot\nabla)(u_\sigma+u_\pi) - 2\Omega\times(u_\sigma+u_\pi) + (\nabla\times b)\times b + f^1),$$

$$-\Pi(((u_\sigma+u_\pi)\cdot\nabla)(u_\sigma+u_\pi) - 2\Omega\times(u_\sigma+u_\pi) + (\nabla\times b)\times b + f^1), \underbrace{0,\dots,0}_{K+1}),$$

$$F_2 = F_2(u_\sigma, u_\pi, b) = \nabla \times ((u_\sigma + u_\pi) \times b) + f^2.$$

In our case, $\mathcal{U}_M = \mathcal{U}_1 \times \operatorname{dom} B_2$ (due to the continuity of the operator B_1).

Theorem 7. *The operator F belongs to $\mathcal{C}^\infty(\mathcal{U}_M; \mathcal{F})$.*

Therefore, the reduction of (1), (2) to (3), (4) is finished. Further, we identify these problems. Now we check the conditions of Theorems 1 and 2.

1.4. Existence Theorem for Unique Solution in Oskolkov Model

By virtue of Theorem 6 and results of Paragraph 3.1 of the work [14], there exists an analytic semigroup $\{U^t : t \in \mathbb{R}_+\}$ of the resolving operators of equation (5), which in this case is represented naturally in the form $U^t = V^t \times W^t$, where $V^t(W^t)$ is the restriction of the operator U^t on $\mathcal{U}_1(\mathcal{U}_2)$. Since B_2 is sectorial, $W^t = \exp(tB_2)$, it follows that the kernel of this semigroup $\mathcal{W}^\circ = \{0\}$, and the image $\mathcal{W}^1 = \mathcal{U}_2$.

Consider the semigroup $\{V^t : t \in \mathbb{R}_+\}$. By virtue of Theorems 4 and 6 and results of Paragraph 3.1 of the work [14], this semigroup is extended to the group $\{V^t : t \in \mathbb{R}\}$. Its kernel $\mathcal{V}^0 = \mathcal{U}_1^{00} \oplus \mathcal{U}_1^{01}$, where $\mathcal{U}_1^{00} = \{0\} \times \{0\} \times \mathbf{H}_p \times \{0\} \times \ldots \times \{0\}(= \ker A_1$ due to Theorem 5), and $\mathcal{U}_1^{01} = \Sigma A_\varkappa^{-1} A_{\varkappa\pi}^{-1}[\mathbf{H}_\pi^2] \times \mathbf{H}_\pi^2 \times \underbrace{\{0\} \times \ldots \times \{0\}}_{K+1}$. Here $A_\varkappa = \mathbb{I} - \varkappa A$, $A_{\varkappa\pi}$ is the restriction of the operator $\Pi A_\varkappa^{-1}$ on $\mathbf{H}_\pi$. It is well known that if $\varkappa^{-1} \notin \sigma(A) \cup \sigma(A_\sigma)$, then the operator $A_{\varkappa\pi} : \mathbf{H}_\pi \to \mathbf{H}_\pi^2$ is a topological linear isomorphism [14]. Denote by $\mathcal{U}_1^1$ the image $\mathcal{V}^1$. Then, due to the strong $(A_1, 1)$-sectoriality of the operator B_1, the space $\mathcal{U}_1$ decomposes into the direct sum of the subspaces $\mathcal{U}_1 = \mathcal{U}_1^{00} \oplus \mathcal{U}_1^{01} \oplus \mathcal{U}_1^1$.

Construct the operator R (see (6), (7)). In our case, $R = B_{10}^{-1} A_{10} \in \mathcal{L}(\mathcal{U}_1^{00} \oplus \mathcal{U}_1^{01})$, where $A_{10}(B_{10})$ is the restriction of the operator $A_1(B_1)$ on $\mathcal{U}_1^{00} \oplus \mathcal{U}_1^{01}$. (The operator B_{10}^{-1} exists due to Theorem 6 and the corresponding results from [14]). By construction, $\ker R = \mathcal{U}_1^{00}$, and the work [35] shows that $\operatorname{im} R = \mathcal{U}_1^{01}$. Then the operator R is bi-splitting. Denote by P_R the space projector $\mathcal{U}_1^{00} \oplus \mathcal{U}_1^{01}$ on $\mathcal{U}_1^{00}$ along $\mathcal{U}_1^{01}$. By construction of the space $\mathcal{U}_M$, the projector P_R belongs to $\mathcal{L}(\mathcal{U}_M^0)$, where $\mathcal{U}_M^0 = \mathcal{U}_M \cap (\mathcal{U}_1^{00} \oplus \mathcal{U}_1^{01})(\equiv \mathcal{U}_1^{00} \oplus \mathcal{U}_1^{01})$. Therefore, the following lemma takes place.

Lemma 1. *Let $\varkappa^{-1} \notin \sigma(A) \cup \sigma(A_\sigma)$. Then the operator R is bi-splitting, and $P_R \in \mathcal{L}(\mathcal{U}_M^0)$.*

Let us introduce the projectors

$$P_k = \operatorname{diag}[\hat{P}_k, 0], \quad Q_k = \operatorname{diag}[\hat{Q}_k, 0], \ k = 0, 1.$$

For a detailed description of these projectors, see [14]. From the results of [14] and from the fact that the kernel $\mathcal{W}^0 = \{0\}$, it follows that $\mathbb{I} - P = (P_0 + P_1) \times O$, $Q = (\mathbb{I} - Q_0 - Q_1) \times \mathbb{I}$, $P : \mathcal{U} \to \mathcal{U}^1$, $Q : \mathcal{F} \to \mathcal{F}^1$. Then, applying the projector $\mathbb{I} - P$ to equation (4) in our situation, we obtain the equations

$$\begin{gathered}\Pi(\nu A(u_\sigma + u_\pi) - ((u_\sigma + u_\pi) \cdot \nabla)(u_\sigma + u_\pi) + \sum_{l=1}^{K} \beta_l \nabla^2 w_l - u_p - 2\Omega \times (u_\sigma + u_\pi) + \\ +(\nabla \times b) \times b + f^1(t)) = 0, \quad B\, u_\pi = 0, \quad B b_\pi = 0.\end{gathered} \tag{12}$$

Hence, by Theorem 1 and the properties of the operator B, we obtain the necessary condition for the existence of the quasi-stationary trajectory $u_\pi \equiv 0$, $b_\pi \equiv 0$, i.e. all solutions to our problem (if such solutions exist) must necessarily belong to the plane $\mathcal{B} = \{u \in \mathcal{U}_M : u_\pi = 0, b_\pi = 0\}$.

Due to the fact that $\Pi u_p = u_p$, from the first equation of (12), we obtain ratio (8), i.e. in our case

$$u_p = \Pi(\nu A u_\sigma - (u_\sigma \cdot \nabla)u_\sigma + \sum_{l=1}^{K} \beta_l \nabla^2 w_l - 2\Omega \times u_\sigma + (\nabla \times b_\sigma) \times b_\sigma + f^1(t)). \quad (13)$$

Lemma 2. *Under the conditions of Lemma 1, any solution to* (1), (2) *belongs to the set*

$$\mathfrak{M} = \{u \in \mathcal{U}_M : u_\pi = 0, b_\pi = 0, u_p = \Pi(\nu A_\sigma - (u_\sigma \cdot \nabla)u_\sigma + \sum_{l=1}^{K} \beta_l \nabla^2 w_l - 2\Omega \times u_\sigma +$$
$$+(\nabla \times b_\sigma) \times b_\sigma) + f^1(t)\}.$$

Remark 4. Condition A$_2$) of Theorem 2 follows from relation (13) for any point $u_0^0 \in \mathcal{U}_M^{00} (\equiv \mathcal{U}_1^{00} \times \{0\})$. Therefore, similarly to [14], we obtain that the set $\mathfrak{M}$ is a simple Banach manifold, which is $\mathcal{C}^\infty$-diffeomorphic to the subspace $\mathcal{U}_1^1 \times \mathcal{U}_2$, and $\mathfrak{M}$ is a candidate for the role of the extended phase space of problem (1), (2) ((2), (13)).

Check conditions (9), (10). Construct the space $\mathcal{U}_\alpha = \mathcal{U}_1 \times \overset{\circ}{W}{}_2^1(D)$. Obviously, this space is an interpolation space for the couple $[\mathcal{U}, \mathcal{U}_M]_\alpha$, and $\alpha = 1/2$. As noted earlier, the semigroup $\{U^t : t \in \mathbb{R}_+\}$ is extended to the group $\{V_1^t : t \in \mathbb{R}\}$ on $\mathcal{U}_1^1$, where V_1^t is the restriction of the operator V^t on $\mathcal{U}_1^1$. By construction, $\mathcal{U}_M^1 = \mathcal{U}_M \cap \mathcal{U}_1^1$, while the operator B_1 is continuous by virtue of Theorem 4, and the semigroup $\{U^t : t \in \mathbb{R}_+\}$ is uniformly bounded. Hence, we obtain the inequality

$$\int_0^\tau \|V_1^t\|_{\mathcal{L}(\mathcal{U}_1^1; \mathcal{U}_M^1)} \, dt \le \text{const} \times \|B_1\|_{\mathcal{L}(\mathcal{U}_1; \mathcal{F}_1)} \int_0^\tau \|V_1^t\|_{\mathcal{L}(\mathcal{U}_1^1)} \, dt < \infty, \ \tau \in \mathbb{R}_+. \quad (14)$$

According to Sobolev inequality [14], the semigroup $\{W^t : \ t \in \bar{\mathbb{R}}_+\}$ satisfies the estimate

$$\int_0^\tau \|W^t\|_{\mathcal{L}(\operatorname{dom} B_2; \overset{\circ}{W}{}_2^1(D))} \, dt < \infty. \quad (15)$$

Let $\mathcal{U}_\alpha^1 = \mathcal{U}_\alpha \cap \mathcal{U}^1$, where $\mathcal{U}^1 = \mathcal{U}_1^1 \times \mathcal{U}_2$. Then, from inequalities (14) and (15), we obtain the following lemma.

Lemma 3. *Under the conditions of Lemma 2, ratio* (9) *holds.*

Now, taking into account condition (10), we find the operator H. The operator H is naturally represented in the form $H = H_1 \otimes H_2$, where $H_1 = A_{11}^{-1}(\mathbb{I} - Q_0 - Q_1)F_1$, and $H_2 \equiv F_2$ (A_{11} is the restriction of the operator A_1 on $\mathcal{U}_1^1$). For the operator H, a statement similar to Theorem 7 for the operator F is true, i.e. $H \in \mathcal{C}^\infty(\mathcal{U}_M^1; \mathcal{U}_\alpha^1)$, where $\mathcal{U}_\alpha^1 = \mathcal{U}_\alpha \cap \mathcal{U}^1$.

Therefore, all the conditions of Theorem 2 are satisfied. Hence, the following statement takes place.

Theorem 8. *Let $\varkappa^{-1} \notin \sigma(A) \cup \sigma(A_\sigma)$. Then, for any u_0 such that $u_0 \in \mathfrak{M}$, and some $T \in \mathbb{R}_+$, there exists the unique solution $u = (u_\sigma, 0, u_p, u_b)$ to problem (1), (2), which is a quasi-stationary trajectory, and $u(t) \in \mathfrak{M}$ for all $t \in (0, T)$.*

2. Thermoconvection Problem for Oskolkov Model

The system of equations

$$\begin{aligned}
&(1-\lambda\nabla^2)\mathbf{v}_t = \nu\nabla^2\mathbf{v} - (\mathbf{v}\cdot\nabla)\mathbf{v} + \sum_{l=1}^{k}\beta_l\nabla^2\mathbf{w}_l - g\mathbf{q}\theta - \mathbf{p} + \mathbf{f},\\
&0 = \nabla(\nabla\cdot\mathbf{v}),\\
&\frac{\partial\mathbf{w}_l}{\partial t} = \mathbf{v} + \alpha_l\mathbf{w}_l,\ \alpha_l\in\mathbb{R}_-,\ l=\overline{1,\ k},\\
&\theta_t = æ\nabla^2\theta - \mathbf{v}\cdot\nabla\theta + \mathbf{v}\cdot\mathbf{q}
\end{aligned} \tag{16}$$

models the evolution of the velocity $\mathbf{v} = (v_1,\ldots,v_n), v_i = v_i(x,t)$, pressure gradient $\mathbf{p} = (p_1,\ldots,p_n)$, $p_i = p_i(x,t)$ and the temperature $\theta = \theta(x,t)$ of an incompressible viscoelastic Kelvin–Voigt fluid of the order $k > 0$ [19]. The parameters $\lambda\in\mathbb{R}$, $\nu\in\mathbb{R}_+$ and $æ\in\mathbb{R}_+$ characterize the elasticity, viscosity and thermal conductivity of the fluid, respectively; $g\in\mathbb{R}_+$ is the acceleration of gravity; the vector $\mathbf{q} = (0,\ldots,0,1)$ is the unit vector in $\mathbb{R}^n$. The parameters $\beta_l\in\mathbb{R}_+$, $l=\overline{1,\ k}$ determine the time of pressure retardation (lagging). The free term $\mathbf{f} = (f_1,\ldots,f_n)$, $f_i = f_i(x,t)$ responds to external influences on the fluid.

Consider the solvability of the first initial-boundary value problem

$$\begin{aligned}
&\mathbf{v}(x,0) = \mathbf{v}_0(x),\quad \mathbf{w}_l(x,0) = \mathbf{w}_{l_0}(x),\\
&\theta(x,0) = \theta_0(x),\ \forall x\in\Omega;\\
&\mathbf{v}(x,t) = 0,\quad \mathbf{w}_l(x,t) = 0,\\
&\theta(x,t) = 0,\ \forall (x,t)\in\partial\Omega\times\mathbb{R}_+,\ l=\overline{1,\ k}
\end{aligned} \tag{17}$$

for the system (16). Here $\Omega\subset\mathbb{R}^n$, $n = 2,3,4$, is the bounded domain with the boundary $\partial\Omega$ of the class C^∞. Problem (16), (17) was studied by professor G.A. Sviridyuk [39] in the particular case $(k = 0,\ f = f(x))$. This problem was at first considered in [25] for an incompressible viscoelastic Kelvin–Voigt fluid of the order $k > 0$ with the nonstationary free term $f(x,t)$, and for an autonomous case in [26].

In [25], the local unique solvability of problem (16), (17) is established. This problem is considered within the framework of the theory of non-autonomous semilinear Sobolev-type equations described above in Section 2. Problem (16), (17) is studied as a concrete interpretation of the abstract Cauchy problem for the specified class of equations (3), (4).

Here, the same approach is used as in the study of the problem of magnetohydrodynamics, that is, problem (16), (17) is reduced to problem (3), (4). To this end, we at first describe the corresponding Banach spaces and the operators acting in them. Then, the fulfillment of all conditions that guarantee the existence of a unique solution to the problem is checked. The solution is a quasistationary semitrajectory. And along the way, the description of the extended phase space of problem (16), (17) is given. Here we confine ourselves to the formulation of the existence and uniqueness theorem for the solution.

Theorem 9. *Let* $\lambda^{-1}\notin\sigma(A)\cup\sigma(A_\sigma)$. *Then, for any* u_0 *such that* $(u_0,0)\in\mathcal{A}^0$ *and some* $T\in\mathbb{R}_+$, *there exists the unique solution* $u\in\mathcal{C}^\infty((0,T);\ \mathcal{U}_M)$ *to problem* (16), (17), *which is a quasi-stationary trajectory, and* $(u(t),t)\in\mathcal{A}^t$.

Under the conditions of Theorem 9, any solution to problem (16), (17) belongs to the set

$$\mathcal{A}^t = \{(u,t): \ u \in \mathcal{U}_M, \ t \in \bar{\mathbb{R}}_+, \quad u_\pi = 0,$$
$$u_p = \Pi(\nu A u_\sigma - (u_\sigma \cdot \nabla) u_\sigma + \sum_{l=1}^{k} \beta_l \nabla^2 w_l - gqu_\theta + f(t))\}.$$

Here the set $\mathcal{A}^t$ is a simple Banach manifold, which is $\mathcal{C}^\infty$-diffeomorphic to the subspace $\mathcal{U}_1^1 \times \mathcal{U}_2$. Moreover, the set $\mathcal{A}^t$ is an extended phase space of problem (16), (17), $\mathcal{U} = \mathcal{U}_1 \times \mathcal{U}_2$, $\mathcal{F} = \mathcal{F}_1 \times \mathcal{F}_2$. The vector u of the space $\mathcal{U}$ has the form $u = \operatorname{col}(u_\sigma, u_\pi, u_p, w_1, \ldots, w_k, u_\theta)$, where $\operatorname{col}(u_\sigma, u_\pi, u_p, w_1, \ldots, w_k) \in \mathcal{U}_1$, and $u_\theta \in \mathcal{U}_2$. Here $u_\sigma = \Sigma v$, $u_\pi = (\mathbb{I} - \Sigma)v = \Pi v$, $u_p = \bar{p}$. The vector $f \in \mathcal{F}$ has a similar form. The operators L and M are defined by the formulas $L = A_1 \otimes A_2$ and $M = B_1 \otimes B_2$. The operator $L \in \mathcal{L}(\mathcal{U}; \mathcal{F})$, and the operator $M: \ \operatorname{dom} M \to \mathcal{F}$ is linear, closed and densely defined, $\operatorname{dom} M = \mathcal{U}_1 \times \operatorname{dom} B_2$.

One can prove that the operator M is strongly $(L,1)$-sectorial. Hence, we arrive at Theorem 9. A detailed description of the spaces $\mathcal{U}$ and $\mathcal{F}$, as well as their subspaces, is presented in [25].

3. Taylor Problem for Oskolkov System

3.1. Problem Statement

The Oskolkov system of equations

$$\begin{cases} (1 - \kappa \nabla^2) v_t = \nu \nabla^2 v - (v \cdot \nabla) v - \nabla p + f, \\ 0 = \nabla \cdot v \end{cases} \qquad (18)$$

models the dynamics of an incompressible viscoelastic Kelvin–Voigt fluid of the zero order [19]. This fluid is the simplest one among non-Newtonian fluids, systematic research of which was started by A.P. Oskolkov.

The function $v = (v_1, v_2, \ldots, v_n)$, $v_i = v_i(x,t)$, $x \in \mathbb{R}^n$ corresponds to the fluid velocity, and the function $p = p(x,t)$ corresponds to the pressure. The function $f = (f_1, f_2, \ldots, f_n)$, $f_i = f_i(x,t)$ characterizes an external influence that is assumed to be known. The real parameters κ and ν characterize the elastic and viscous properties of the fluid, respectively. It is known that $\nu > 0$. As for κ, experimental data [1] and theoretical studies [49] support the fundamental unboundedness of the domain of definition of this parameter.

Taylor problem for system (18) simulates a situation when the viscoelastic incompressible Kelvin–Voigt fluid occupies the space between two rotating coaxial cylinders of infinite length [13].

In this situation, the bounded domain $\Omega \subset \mathbb{R}^n$, $n = 2, 3, 4$ (with a piecewise-smooth boundary) is chosen such that the periodicity condition is satisfied (i.e. $v(x,t)|_{\partial\Omega \cap \alpha} = v(x,t)|_{\partial\Omega \cap \beta}$, $\partial\Omega \cap (\alpha \cup \beta) = \partial_1 \Omega \ \ \forall t \in \mathbb{R}$) on a part of the boundary $\partial_1 \Omega$ (that belongs, for example, for $n = 3$, to two planes α and β perpendicular to the cylinder axis) of the domain Ω.

In addition, we choose some stationary solution $\tilde{v} = \tilde{v}(x)$ of system (18) satisfying the periodicity condition on $\partial_1 \Omega$ and the inhomogeneous Dirichlet conditions (for example, Couette flow) on $\partial_2 \Omega = \partial\Omega \backslash \partial_1 \Omega$. Then, we investigate dynamics of deviation of $v = v(x,t)$ from this stationary solution caused by the initial condition. Therefore, system (18) takes the form

$$\begin{cases} (1-\kappa\nabla^2)v_t = \nu\nabla^2 v - (v\cdot\nabla)\tilde{v} - (\tilde{v}\cdot\nabla)v - (v\cdot\nabla)v - \nabla p, \\ 0 = \nabla\cdot v. \end{cases} \tag{19}$$

Consider the Taylor problem

$$\begin{aligned} &v(x,0) = v_0(x),\ \forall x\in\Omega, \\ &v(x,t) = 0,\ \forall(x,t)\in\partial_2\Omega\times\mathbb{R}, \\ &v(x,t) \text{ that satisfies the periodicity condition on } \partial_1\Omega\times\mathbb{R} \end{aligned} \tag{20}$$

for system (18).

Similar to the previous problems, we move from system (19) to the system

$$\begin{cases} (1-\kappa\nabla^2)v_t = \nu\nabla^2 v - (v\cdot\nabla)\tilde{v} - (\tilde{v}\cdot\nabla)v - (v\cdot\nabla)v - \bar{p}, \\ 0 = \nabla(\nabla\cdot v). \end{cases} \tag{21}$$

Replacement $\bar{p} = \nabla p$ is explained by the fact that, in most hydrodynamic problems, the consideration of the pressure gradient is preferable to the consideration of the pressure. Although the gradient operator, generally speaking, has a kernel spanned by a constant, the study of system (21) gives no new solutions other than solutions of system (19), since we consider spaces of functions given in a bounded domain and satisfying the adhesion conditions on the boundary of the domain. This follows from the Gauss–Ostrogradsky theorem.

In this section, we study the local unique solvability of problem (20), (21) based on the theory of semilinear Sobolev-type equations [33]. Earlier, Taylor problem for an incompressible viscoelastic Kelvin–Voigt fluid was studied in another aspect in [37].

3.2. Abstract Cauchy Problem

Let $\mathcal{U}$ and $\mathcal{F}$ be Banach spaces, the operators $L\in\mathcal{L}(\mathcal{U};\mathcal{F})$ and $M\in\mathcal{C}^\infty(\mathcal{U};\mathcal{F})$, the function $f:\mathbb{R}\to\mathcal{F}$. Consider the Cauchy problem

$$u(0) = u_0 \tag{22}$$

for the *semilinear nonstationary Sobolev-type equation*

$$L\dot{u} = M(u) + f. \tag{23}$$

It is known that the linear operator $L:\mathcal{U}\to\mathcal{F}$ is called *bi-splitting*, if its kernel $\ker L$ and the image $\operatorname{im} L$ are complemented in the spaces $\ker L$ and $\mathcal{F}$, respectively. Let the operator L be bi-splitting. Denote by $M'_{u_0}\in\mathcal{L}(\mathcal{U};\mathcal{F})$ the Frechet derivative of the operator M at the point $u_0\in\mathcal{U}$ and consider the chain of M'_{u_0}-associated vectors of the operator L, which we choose from some complement $\operatorname{coim} L = \mathcal{U}\ominus\ker L$ to the kernel $\ker L$. Let us introduce the following condition.

(A1) Regardless of the choice of $\operatorname{coim} L$, any chain of M'_{u_0}-associated vectors of any vector $\varphi\in\ker L\setminus\{0\}$ contains exactly p elements.

Denote by $\tilde{L}$ the restriction of the operator $\operatorname{coim} L$. By virtue of the Banach theorem on a closed graph, the operator $\tilde{L}:\operatorname{coim} L\to\operatorname{im} L$ is a toplinear isomorphism. Assume that $\mathcal{U}_0^0 = \ker L$ and construct the sets $\mathcal{U}_q^0 = \tilde{A}^q[\mathcal{U}_0^0]$, $q = 1,2,\ldots,p$, where $\tilde{A} = \tilde{L}^{-1}M'_{u_0}$.

Obviously, the sets $\mathcal{U}_q^0 \subset \operatorname{coim} L$ are linear spaces, therefore, the image $\mathcal{F}_p^0 = M'_{u_0}[\mathcal{U}_p^0]$ is also a linear space, and $\mathcal{F}_p^0 \cap \operatorname{im} L = \{0\}$ (if (A1) holds).

Let us introduce into consideration one more condition.

(A2) $\mathcal{F}_p^0 \oplus \operatorname{im} L = \mathcal{F}$.

Lemma 4. *Let the operators* $L \in \mathcal{L}(\mathcal{U};\mathcal{F})$, $M \in \mathcal{C}^\infty(\mathcal{U};\mathcal{F})$, *where* L *is a bi-splitting operator, and conditions* (A1) *and* (A2) *are satisfied. Then equation* (23) *is equivalent to the corresponding system [33].*

Let us begin searching for a solution to problem (22), (23).

Definition 3. *By a solution to problem* (22), (23) *we mean the vector-function*

$$u \in \mathcal{C}^\infty((-t_0;t_0);\mathcal{U}),\ t_0 = t_0(u_0) > 0,$$

satisfying equation (23) *and condition* (22).

On this way, we need to overcome two difficulties. First, as it is well known [36, 37], solutions to problem (22), (23) do not exist for all $u_0 \in \mathcal{U}$. Second, even if a solution to problem (22), (23) exists, the solution can be not unique [40]. In order to overcome these difficulties, we introduce two definitions.

Definition 4. *The set* $\mathcal{B}^t \subset \mathcal{U} \times \mathbb{R}$ *is said to be the extended phase space of equation* (23), *if, for any point* $u_0 \in \mathcal{U}$ *such that* $(u_0, 0) \in \mathcal{B}^0$, *there exists a unique solution to problem* (22), (23), *and* $(u(t), t) \in \mathcal{B}^t$.

Remark 5. If $\mathcal{B}^t = \mathcal{B} \times \mathbb{R}$, where $\mathcal{B} \subset \mathcal{U}$, then the set $\mathcal{B}$ is called the *phase space* of equation (23) [36, 37, 40].

Definition 5. *The solution* $u = u(t)$ *to problem* (22), (23), *for which* $L\dot{u}^0 \equiv 0\ \forall t \in (-t_0;t_0)$, *where* $u^0 = Pu$, *is called the quasistationary trajectory of equation* (23).

In order to separate quasi-stationary trajectories from the set of possible solutions to problem (22), (23), we impose two conditions.

(A3) $f_q^0(t) \equiv 0\ \forall t \in \mathbb{R},\ q = 1, 2, \ldots, p.$

Consider the set $\tilde{\mathcal{U}} = \{u \in \mathcal{U} : u_q^0 = \text{const},\ q = 1,\ 2, \ldots, p\}$. It is easy to see that $\tilde{\mathcal{U}}$ is a complete affine manifold modeled by the subspace $\mathcal{U}_0^0 \oplus \mathcal{U}^1$. Let the point $u_0 \in \tilde{\mathcal{U}}$. Denote by $\mathcal{O}_{u_0}$ some neighborhood $\mathcal{O}_{u_0} \subset \tilde{\mathcal{U}}$ of the point u_0.

(A4) $F_q(u) \equiv 0\ \forall u \in \mathcal{O}_{u_0},\ q = 1, 2, \ldots, p.$

Theorem 10. *Let*
(i) *the conditions of Lemma 4 be satisfied;*
(ii) *the point* $(u_0, 0) \in \mathcal{B}^0$, *where* $\mathcal{B}^t = \{(u,t) \in \tilde{\mathcal{U}} \times \mathbb{R} :\ Q_0(M(u) + f(t)) = 0\}$;
(iii) *the vector-function* $f \in \mathcal{C}^\infty(\mathbb{R};\mathcal{F})$;
(iv) *conditions* (A3), (A4) *be satisfied.*

Then there exists a unique solution to problem (22), (23), *which is a quasi-stationary trajectory, and* $(u(t), t) \in \mathcal{B}^t\ \forall t \in (-t_0;t_0)$.

3.3. Reduction to Abstract Cauchy Problem and Existence Theorem for Unique Solution

Let us reduce problem (20), (21) to problem (22), (23). To this end, we set

$$\mathcal{U} = \mathbf{H}_\sigma^2 \times \mathbf{H}_\pi^2 \times \mathbf{H}_p, \quad \mathcal{F} = \mathbf{H}_\sigma \times \mathbf{H}_\pi \times \mathbf{H}_p, \tag{24}$$

where $\mathbf{H}_\sigma$ is a closure of the lineal $\{v \in (\mathcal{C}^\infty(\Omega))^n : \nabla \cdot v = 0;\ \operatorname{supp} v \cap \partial_2\Omega = \emptyset$ and the periodicity condition is satisfied on $\partial_1\Omega,\ \partial_1\Omega \cup \partial_2\Omega = \partial\Omega\}$ in the norm $\mathbf{L}^2(\Omega) = (L^2(\Omega))^n$, $\mathbf{H}_\pi = \mathbf{H}_\sigma^\perp$, $\mathbf{H}_p = \mathbf{H}_\pi$. Denote the orthoprojector by $\Sigma : \mathbf{L}^2(\Omega) \to \mathbf{H}_\sigma$. Then $\Sigma \in \mathcal{L}((W_2^2(\Omega) \cap \overset{\circ}{W}{}_2^1(\Omega))^n$. Let $\operatorname{im}\Sigma = \mathbf{H}_\sigma^2, \ker\Sigma = \mathbf{H}_\pi^2$. The element $u \in \mathcal{U}$ has the form $u = (u_\sigma, u_\pi, u_p)$, where $u_\sigma = \Sigma v, u_\pi = \Pi v, u_p = \bar{p}$.

The operators $L, M : \mathcal{U} \to \mathcal{F}$ are defined by the formulas

$$L = \begin{pmatrix} \Sigma A_\kappa \Sigma & \Sigma A_\kappa \Pi & O \\ \Pi A_\kappa \Sigma & \Pi A_\kappa \Pi & O \\ O & O & O \end{pmatrix}, \tag{25}$$

where $\Pi = \mathbb{I} - \Sigma, \quad A_\kappa = 1 - \kappa\nabla^2$;

$$M(u) = \begin{pmatrix} \Sigma B(u_\sigma + u_\pi) \\ \Pi B(u_\sigma + u_\pi) - u_p \\ C(u_\sigma + u_\pi) \end{pmatrix}, \tag{26}$$

where $B(u_\sigma+u_\pi) = \nu\nabla^2(u_\sigma+u_\pi) - ((u_\sigma+u_\pi)\cdot\nabla)\tilde{v} - (\tilde{v}\cdot\nabla)(u_\sigma+u_\pi) - ((u_\sigma+u_\pi)\cdot\nabla)(u_\sigma+u_\pi)$, $C(u_\sigma + u_\pi) = \nabla(\nabla \cdot (u_\sigma + u_\pi))$.

Lemma 5. *Let the spaces* $\mathcal{U}$, $\mathcal{F}$ *be defined by formulas* (24), *where* $n = 2, 3, 4$, *and the operators* $L, M : \mathcal{U} \to \mathcal{F}$ *be defined by formulas* (25), (26). *Then*
(i) *the operator* $L \in \mathcal{L}(\mathcal{U}; \mathcal{F})$, *and if* $\kappa^{-1} \notin \sigma(-\nabla^2)$, *then* $\ker L = \{0\} \times \{0\} \times \mathbf{H}_p$, $\operatorname{im} L = \mathbf{H}_\sigma \times \mathbf{H}_\pi \times \{0\}$;
(ii) *the operator* $M \in \mathcal{C}^\infty(\mathcal{U}; \mathcal{F})$.

This completes the reduction of problem (20), (21) to problem (22), (23). Next, we check that conditions (A1) – (A4) are satisfied, see [33] for details. In order to check condition (A1), we denote by $A_{\kappa\sigma}$ the restriction of the operator $\Sigma A_\kappa \Sigma$ on $\mathbf{H}_\sigma^2$.

Lemma 6. *Let the conditions of Lemma 5 be satisfied, moreover* $\ker A_{\kappa\sigma} = \{0\}$. *Then each vector* $\varphi \in \ker L \setminus \{0\}$ *has exactly one* M'_u*-connected vector regardless of the point* $u \in \mathcal{U}$.

Lemma 6 implies that condition (A1) is satisfied, and $p = 1$. As a result of checking conditions (A1) – (A4), we establish the following theorem.

Theorem 11. *[33] Let the conditions of Lemma 6 be satisfied. Let* $u_0 \in \mathcal{B}$. *Then, for some* $t_0 = t_0(u_0)$, *there exists a unique solution* $u = (u_\sigma, 0, \bar{p}) \in \mathcal{C}^\infty((-t_0, t_0); \mathcal{B})$ *to problem* (20), (21), *which is a quasi-stationary trajectory.*

Here $\mathcal{B} = \{u \in \tilde{\mathcal{U}} :\ A_{\kappa\pi}^{-1}\Pi A_\kappa^{-1} B(u_\sigma) = u_p,\ u_\pi = 0\}$.

Remark 6. In the dynamic case, non-stationary Oskolkov models of the incompressible viscoelastic fluids were considered in [23,24]. Linearized Oskolkov models of different orders were studied in [27–31]. The monograph [14] is devoted to the study of Oskolkov models in the autonomous case. Other directions in the theory of Sobolev-type equations and its applications are outlined in [2, 3, 21, 22, 38, 44, 47, 51].

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МОДЕЛИ ОСКОЛКОВА И УРАВНЕНИЯ СОБОЛЕВСКОГО ТИПА

Т.Г. Сукачева[1,2]
[1]Новгородский государственный университет имени Ярослава Мудрого, г. Великий Новгород, Российская Федерация
[2]Южно-Уральский государственный университет, г. Челябинск, Российская Федерация

Данная статья посвящена обзору работ, выполненных автором совместно со своими учениками и посвященных исследованию различных моделей Осколкова. Их отличительной особенностью является использование полугруппового подхода, лежащего в основе метода фазового пространства, широко применяемого в теории уравнений соболевского типа. Приведены различные модели несжимаемой вязкоупругой жидкости, описываемые уравнениями Осколкова. В качестве примеров рассмотрены вырожденная задача магнитогидродинамики, задача термоконвекции и задача Тейлора. Разрешимость соответствующих начально-краевых задач исследуется в рамках теории уравнений соболевского типа, основанной на теории относительно p-секториальных операторов и вырождающихся полугрупп операторов. Доказана теорема существования единственного решения, являющегося квазистационарной полутраекторией, и получено описание расширенного фазового пространства. Основы теории разрешимости уравнений соболевского типа были заложены профессором Г.Свиридюком. Затем эта теория вместе с различными приложениями была успешно развита его последователями.

Ключевые слова: системы Осколкова; уравнения соболевского типа; фазовое пространство; несжимаемая вязкоупругая жидкость.

Тамара Геннадьевна Сукачева, доктор физико-математических наук, профессор, кафедра «Алгебра и геометрия», Новгородский государственный университет имени Ярослава Мудрого (г. Великий Новгород, Российская Федерация); научно-исследовательская лаборатория «Неклассические уравнения математической физики», Южно-Уральский государственный университет (г. Челябинск, Российская Федерация), tamara.sukacheva@novsu.ru.

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