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SOBOLEV TYPE EQUATIONS IN SPACES OF DIFFERENTIAL FORMS ON RIEMANNIAN MANIFOLDS WITHOUT BOUNDARY

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The article contains a review of the results obtained by the author both independently and in collaboration with other members of the Chelyabinsk scientific school founded by G.A. Sviridyuk and devoted to Sobolev-type equations in specific spaces, namely the spaces of differential forms defined on some Riemannian manifold without boundary. Sobolev type equations are nonclassical equations of mathematical physics and are characterized by an irreversible operator at the highest derivative. In our spaces, we need to use special generalizations of operators to the space of differential forms, in particular, the Laplace operator is replaced by its generalization, the Laplace–Beltrami operator. We consider specific interpretations of equations with the relatively bounded operators: linear Barenblatt–Zheltov–Kochina, linear and semilinear Hoff, linear Oskolkov ones. For these equations, we investigate the solvability of the Cauchy, Showalter–Sidorov and initial-final value problems in different cases. Depending on the choice of the type of equation (linear or semi-linear), we use the corresponding modification of the phase space method. In the spaces of differential forms, in order to use this method based on domain splitting and the actions of the corresponding operators, the basis is the Hodge–Kodaira theorem on the splitting of the domain of the Laplace-Beltrami operator.

Keywords: Sobolev-type equations; phase space method; differential forms; Riemannian manifold without boundary.

Dedicated to the 70th anniversary of the Teacher
Professor Georgy Anatolyevich Sviridyuk

Introduction

Consider the following equations:

– the linear Barenblatt–Zheltov–Kochina equation [1]

$$(\lambda - \Delta)u_t = \alpha\Delta u, \tag{1}$$

which is a model of dynamics of a fluid filtering in a fractured-porous environment;

– the Oskolkov linear equation [29]

$$(1 - \kappa\Delta)\Delta\varphi_t = \nu\Delta^2\varphi, \tag{2}$$

which is a model of flow of a viscous-elastic incompressible zero-order Kelvin–Voigt fluid in the first approximation;

– the semilinear Hoff equation [5, 24]

$$(\lambda - \Delta)u_t = \alpha u + \beta u^3, \tag{3}$$

which is a model of buckling of an I-beam.

In the functional spaces $\mathfrak{U}$, $\mathfrak{F}$ chosen by us, (1), (2) are reduced [22] to the linear equation of Sobolev type

$$L\dot{u} = Mu \tag{4}$$

with the irreversible operator L, while equation (3) is reduced to the semilinear equation of Sobolev type

$$L\dot{u} = Mu + N(u). \tag{5}$$

For these equations, the Cauchy problem [28]

$$u(0) = u_0, \tag{6}$$

the Showalter–Sidorov problem [30]

$$P(u(0) - u_0) = 0, \tag{7}$$

and the initial-final value problem [32]

$$P_0(u(0) - u_0) = 0, P_T(u(T) - u_T) = 0 \tag{8}$$

were considered.

Introduction indicates the range of equations and systems that are included in our review. Section 1 ("Introductory Information") is divided into two subsections. Subsection 1.1 ("Terminology of Sobolev Type Equations and Phase Space Method") contains information from the theory of Sobolev type equations about the relatively bounded operators and the phase space method developed by G.A. Sviridyuk and T.G. Sukacheva. In Subsection 1.2 ("Spaces of Differential Forms and Splitting of Action of Abstract Operators"), we construct spaces in which solvability is studied, namely, orthogonal to harmonic smooth differential k-forms defined on a n-dimensional connected smooth compact oriented Riemannian manifold without boundary [31]. In each of subsections of Section 2 ("Investigation of Linear Equations"), one of the three linear equations is analyzed. Section 3 ("Investigation of Semilinear Equations") describes the phase space of the semilinear equation in Subsection 3.1 ("Phase Space for Semilinear Equations of Sobolev Type") and the structure of the phase space containing a solution to the Cauchy problem for the semilinear Hoff equation in Subsection 3.2 ("Semilinear Hoff Equation"). In Conclusion, we describe other areas of research of Sobolev-type equations, which were considered earlier or are of interest for future study in spaces of differential forms.

1. Introductory Information

1.1. Terminology of Sobolev Type Equations and Phase Space Method

Let $\mathfrak{U}$ and $\mathfrak{F}$ be Banach spaces, the operators $L,\ M \in \mathcal{L}(\mathfrak{U};\mathfrak{F})$. Consider the L-resolvent set $\rho^L(M) = \{\mu \in \mathbb{C} : (\mu L - M)^{-1} \in \mathcal{L}(\mathfrak{F};\mathfrak{U})\}$ and the L-spectrum $\sigma^L(M) = \mathbb{C} \setminus \rho^L(M)$ of the operator M. If the L-spectrum $\sigma^L(M)$ of the operator M is bounded, then the operator M is said to be (L,σ)-bounded. If the operator M is (L,σ)-bounded, then there exist the projectors

$$P = \frac{1}{2\pi i}\int\limits_{\gamma} R^L_\mu(M)d\mu \in \mathcal{L}(\mathfrak{U}), \quad Q = \frac{1}{2\pi i}\int\limits_{\gamma} L^L_\mu(M)d\mu \in \mathcal{L}(\mathfrak{F}).$$

Here $R^L_\mu(M) = (\mu L - M)^{-1}L$ and $L^L_\mu(M) = L(\mu L - M)^{-1}$ are the *right* and *left L-resolvents* of the operator M, respectively, while the closed contour $\gamma \subset \mathbb{C}$ bounds a domain that contains $\sigma^L(M)$. Let $\mathfrak{U}^0$ ($\mathfrak{U}^1$) $= \ker P$ (im$P$), $\mathfrak{F}^0$ ($\mathfrak{F}^1$) $= \ker Q$ (im$Q$) and denote by L_k (M_k) the restriction of the operator L (M) onto $\mathfrak{U}^k$, $k = 0, 1$.

Theorem 1. [22] *Let the operator M be (L, σ)-bounded, then*
(i) the operators L_k (M_k) $\in \mathcal{L}(\mathfrak{U}^k; \mathfrak{F}^k)$, $k = 0, 1$;
(ii) there exist the operators $M_0^{-1} \in \mathcal{L}(\mathfrak{F}^0; \mathfrak{U}^0)$ and $L_1^{-1} \in \mathcal{L}(\mathfrak{F}^1; \mathfrak{U}^1)$.

Corollary 1. [22] *Let the operator M be (L, σ)-bounded, then*

$$(\mu L - M)^{-1} = -\sum_{k=0}^{\infty} \mu^k S^{k-1} L_1^{-1} Q + \sum_{k=1}^{\infty} \mu^{-k} H^k M_0^{-1}(\mathbb{I} - Q),$$

the operator $H = L_0^{-1} M_0 \in \mathcal{L}(\mathfrak{U}^0)$, $S = L_1^{-1} M_1 \in \mathcal{L}(\mathfrak{U}^1)$.

Hereinafter, the (L, σ)-bounded operator M is said to be (L, p)-*bounded,* $p \in \{0\} \cup \mathbb{N}$, if ∞ is a removable singular point ($H \equiv \mathbb{O}$, $p = 0$) or a pole of the order $p \in \mathbb{N}$ (i.e. $H^p \neq \mathbb{O}$, $H^{p+1} \equiv \mathbb{O}$) of the *L-resolvent* $(\mu L - M)^{-1}$ of the operator M. We consider the vector-function $u \in C^1(\mathbb{R}; \mathfrak{U})$ to be a *solution* to equation (4), if when substituting u into (4), this equation turns into an identity. A solution $u = u(t)$ to equation (4) is said to be a *solution to the Cauchy problem*

$$u(0) = u_0 \tag{9}$$

for equation (4), if equality (9) holds for some $u_0 \in \mathfrak{U}$.

Definition 1. *The set $\mathfrak{P} \subset \mathfrak{U}$ is said to be a phase space of equation (4), if*
(i) any solution $u = u(t)$ to equation (4) belongs to $\mathfrak{P}$ pointwise, i.e. $u(t) \in \mathfrak{P}$ for all $t \in \mathbb{R}$;
(ii) for any $u_0 \in \mathfrak{P}$ there exists a unique solution $u \in C^1(\mathbb{R}; \mathfrak{U})$ to Cauchy problem (9) for equation (4).

Theorem 2. [22] *Let the operator M be (L, p)-bounded, $p \in \{0\} \cup \mathbb{N}$. Then the phase space of equation (4) is the subspace $\mathfrak{U}^1$.*

Note that if the operator $L^{-1} \in \mathcal{L}(\mathfrak{F}; \mathfrak{U})$ exists, then the phase space of equation (4) is the space $\mathfrak{U}$.

1.2. Spaces of Differential Forms and Splitting of Action of Abstract Operators

Let $\mathcal{M}$ be a smooth compact oriented Riemannian manifold without boundary with the local coordinates $x_1, x_2, ..., x_n$. Denote the spaces of smooth differential k-forms, $k = 0, 1, 2, ..., n$, by $H_k = H_k(\mathcal{M})$.

The differential forms have the form

$$\chi_{i_1, i_2, ..., i_k}(t, x_1, x_2, ..., x_n) = \sum_{|i_1, i_2, ..., i_k| = k} a_{i_1, i_2, ..., i_k}(t, x_{i_1}, x_{i_2}, ..., x_{i_k}) dx_{i_1} \wedge dx_{i_2} \wedge ... \wedge dx_{i_k},$$

where $a_{i_1, i_2, ..., i_k}(t, x_{i_1}, x_{i_2}, ..., x_{i_k})$ are coefficients depending on time as well, and $|i_1, i_2, ..., i_k|$ is a multi-index.

The spaces H_k are endowed with the standard scalar product

$$(\xi, \varepsilon)_0 = \int_{\mathcal{M}} \xi \wedge *\varepsilon, \quad \xi, \varepsilon \in H_k. \tag{10}$$

Here $*$ is the Hodge operator and $\wedge$ is the operator of the inner multiplication of k-forms.

Completing the space H_k by continuity in the norm $\|\cdot\|_0$ corresponding to scalar product (10), we obtain the space $\mathfrak{H}_k^0$. Introducing scalar products in spaces of one or twice differentiable k-forms and completing spaces according to the norms corresponding to these scalar products, we construct the spaces $\mathfrak{H}_k^1$, $\mathfrak{H}_k^2$. There exist continuous embeddings of the resulting Hilbert spaces

$$\mathfrak{H}_k^2 \subseteq \mathfrak{H}_k^1 \subseteq \mathfrak{H}_k^0.$$

In these spaces, we can use a generalization of the Laplace–Beltrami operator

$$\Delta u = (d\delta + \delta d)u,$$

where d is the operator of external multiplication of differential forms, and the operator $\delta = *d*$ is the adjoint operator of d.

For the resulting spaces, a generalization of the Hodge–Kodaira theorem takes place.

Theorem 3. [31] *Consider the spaces* $\mathfrak{H}_k^l, l = 0, 1, 2$. *Then*

$$\mathfrak{H}_k^l = \mathfrak{H}_{kd}^l \oplus \mathfrak{H}_{k\delta}^l \oplus \mathfrak{H}_{k\Delta}^l, l = 0, 1, 2.$$

Here $\mathfrak{H}_{kd}$, $\mathfrak{H}_{k\delta}$ *and* $\mathfrak{H}_{k\Delta}$ *are potential, solenoidal, and harmonic forms, respectively.*

Corollary 2. [18] *Under the conditions of Theorem 3, there exists the decomposition* $\mathfrak{H}_k^l = (\mathfrak{H}_{k\Delta}^l)^\perp \oplus \mathfrak{H}_{k\Delta}^l, l = 0, 1, 2$.

As the main space in which we study the solvability of the corresponding problems, we take (see Subsection 1.1) $\mathfrak{U} = (\mathfrak{H}_{k\Delta}^2)^\perp$.

The spectrum of the Laplace–Beltrami operator $\sigma(\Delta)$ is discrete, finite multiple, and condenses only to $+\infty$. Next, $\{\lambda_l\}$ is a sequence of eigenvalues of the Laplace–Beltrami operator numbered non-increasingly, taking into account their multiplicity, and $\{\varphi_l\}$ is the corresponding sequence of orthonormal (in sense of $\mathfrak{U}$) eigenfunctions.

2. Investigation of Linear Equations

2.1. Barenblatt–Zheltov–Kochina Equation

Consider the Barenblatt–Zheltov–Kochina equation $(\lambda - \Delta)u_t = \alpha\Delta u$ in the space of differential forms $\mathfrak{U} = (\mathfrak{H}_{k\Delta}^2)^\perp$ from Subsection 1.2. For fixed $\alpha, \lambda \in \mathbb{R}$, introduce the operators

$$L = (\lambda + \Delta),\ M = \alpha\Delta. \tag{11}$$

Remark 1. Hereinafter, Δ is the Laplace–Beltrami operator generalizing the ordinary Laplace operator up to sign. Therefore, the sign on the right side in brackets changes to $'+'$, and the sign on the left side goes into the coefficient of the Laplace–Beltrami operator.

We obtain the linear equation of Sobolev type

$$L\dot{u} = Mu. \tag{12}$$

The initial Cauchy condition has the form

$$u\,(0) = u_0. \tag{13}$$

Lemma 1. [26] *For any $\alpha, \lambda \in \mathbb{R} \setminus \{0\}$, the operator M is (L,p)-bounded with $p = 0$.*

Based on the sequences from Subsection 1.2, construct a projector $P \in \mathcal{L}(\mathfrak{U})$ onto the phase space $\mathfrak{U}^1 \subseteq \mathfrak{U}$:

$$P = \begin{cases} \mathbb{I}, \ \lambda \neq \lambda_l \text{ for all } l \in \mathbb{N}; \\ \mathbb{I} - \sum\limits_{\lambda=\lambda_l} \langle \cdot, \varphi_j \rangle \varphi_l, \text{if } \lambda = \lambda_l. \end{cases}$$

Theorem 4. [26] *For any $\lambda, \alpha \in \mathbb{R} \setminus \{0\}$ and $u_0 \in \mathfrak{U}^1$, there exists a unique solution $u = u(t)$ to problem (12), (13), which has the form*

$$u(t) = \sum_{l=1}^{\infty}{}' \left[exp\left(\frac{\alpha\lambda_l}{\lambda + \lambda_l} t\right) (u_0, \varphi_l)_0 \varphi_l \right]. \tag{14}$$

Here, the prime at the sum sign means the absence of terms for which $\lambda = \lambda_l$.

If we consider the inhomogeneous equation

$$L\dot{u} = Mu + f \tag{15}$$

with the Showalter–Sidorov initial condition

$$\left[R_\alpha^L(M)\right]^{p+1} (u(0) - u_0) = 0, \tag{16}$$

then we arrive at the following theorem.

Theorem 5. [26] *For any $\lambda \in \mathbb{R} \setminus \{0\}$, $\alpha \in \mathbb{R} \setminus \{0\}$, $f \in \mathfrak{F}$ and $u_0 \in \mathfrak{U}^1$, there exists a unique solution $u = u(t)$ to problem (15), (16).*

2.2. Oskolkov Equation

Consider the linear Oskolkov equation $(1-\kappa\Delta)\Delta\varphi_t = \nu\Delta^2\varphi$ in the space of differential forms $\mathfrak{U} = (\mathfrak{H}^2_{k\Delta})^\perp$ from Subsection 1.2. For fixed coefficients $\nu, \kappa \in \mathbb{R} \setminus \{0\}$, introduce operators taking into account Remark 1:

$$L = (\lambda + \Delta), \ M = \alpha\Delta. \tag{17}$$

Let $u = \Delta\varphi$, then we arrive at the linear equation of Sobolev type

$$L\dot{u} = Mu. \tag{18}$$

The initial Cauchy condition has the form

$$u(0) = u_0. \tag{19}$$

The operator L constructed above is Fredholm and the following lemma takes place.

Lemma 2. [26] *For any $\alpha, \lambda \in \mathbb{R} \setminus \{0\}$, the operator M is (L,p)-bounded with $p = 0$.*

Based on the sequences from Subsection 1.2, construct a projector $P \in \mathcal{L}(\mathfrak{U})$ onto the phase space $\mathfrak{U}^1$:

$$P = \begin{cases} \mathbb{I}, \ \lambda \neq \lambda_l \text{ for all } l \in \mathbb{N}; \\ \mathbb{I} - \sum\limits_{\lambda=\lambda_l} \langle \cdot, \varphi_j \rangle \varphi_l, \text{if } \lambda = \lambda_l. \end{cases}$$

Theorem 6. [26] *For any* $\lambda, \alpha \in \mathbb{R} \setminus \{0\}$ *and* $u_0 \in \mathfrak{U}^1$, *there exists a unique solution* $u = u(t)$ *to problem (18), (19), which has the form*

$$u(t) = \sum_{l=1}^{\infty}{}' \left[exp\left(\frac{\alpha\lambda_l}{\lambda + \lambda_l} t\right) (u_0, \varphi_l)_0 \varphi_l \right]. \tag{20}$$

Here, the prime at the sum sign means the absence of terms for which $\lambda = \lambda_l$.

2.3. Linear Hoff Equation

Consider the linear Hoff equation $(\lambda - \Delta)u_t = \alpha u$ in the space of differential forms $\mathfrak{U} = (\mathfrak{H}^2_{k\Delta})^{\perp}$ from Subsection 1.2. For fixed coefficients $\lambda, \alpha \in \mathbb{R} \setminus \{0\}$, we introduce the operators taking into account Remark 1:

$$L = (\lambda + \Delta),\ M = \alpha\Delta. \tag{21}$$

The initial-final value conditions have the form

$$P_0(u(0) - u_0) = 0, P_T(u(T) - u_T) = 0. \tag{22}$$

We arrive at the linear equation of Sobolev type

$$L\dot{u} = Mu. \tag{23}$$

The operator L constructed above is Fredholm and the following lemma takes place.

Lemma 3. [33] *For all* $\alpha, \lambda \in \mathbb{R} \setminus \{0\}$, *the operator* M *is* (L, p)*-bounded with* $p = 0$.

Due to (21) and the form of the spectrum of the Laplace–Beltrami operator as a sequence from Subsection 1.2, the L-spectrum of the operator M has the form

$$\sigma^L(M) = \left\{ \mu_l = \frac{\alpha}{\lambda + \lambda_l}, l \in \mathbb{N} \right\}.$$

Let the L-spectrum of the operator M be represented as $\sigma^L(M) = \sigma^L_0(M) \cup \sigma^L_T(M)$, where $\sigma^L_0(M)$ is a nonempty set (such a representation is ambiguous). In this case, we require the existence of a closed contour $\gamma_1 \in \mathbb{C}$ bounding a domain D_1 such that $\sigma^L_T(M) \subset D_1$ and $D_1 \cap \sigma^L_T(0)$ is an empty set. Then there exist the relatively spectral projectors $P_0 = \sum\limits_{\mu_l \in \sigma^L_0(M)} (\cdot, \varphi_l)_0 \varphi_l$ and $P_T = \sum\limits_{\mu_l \in \sigma^L_T(M)} (\cdot, \varphi_l)_0 \varphi_l$, while conditions (22) have the form

$$\sum_{\mu_l \in \sigma^L_0(M)} (u(0) - u_0, \varphi_l)_0 \varphi_l = 0, \sum_{\mu_l \in \sigma^L_T(M)} (u(T) - u_T, \varphi_l)_0 \varphi_l = 0. \tag{24}$$

By virtue of Lemma 3, we have

Theorem 7. [33] *For any* $\alpha \in \mathbb{R} \setminus \{0\}, \lambda \neq \lambda_l$, *initial-final value problem (24) for equation (23) has a unique solution of the form*

$$u(t) = \sum_{\mu_l \in \sigma^L_0(M)} exp(\mu_l t)(u(0), \varphi_l)_0 \varphi_l + \sum_{\mu_l \in \sigma^L_T(M)} exp(\mu_l (t - T))(u(T), \varphi_l)_0 \varphi_l. \tag{25}$$

3. Investigation of Semilinear Equations

3.1. Phase Space for Semilinear Equations of Sobolev Type

Let $\mathfrak{U}$ and $\mathfrak{F}$ be Banach spaces, the operators $L, M \in \mathcal{L}(\mathfrak{U}, \mathfrak{F})$, while the operator $N \in C^\infty(\mathfrak{U}, \mathfrak{F})$. Consider the semilinear equation of Sobolev type

$$L\dot{u} = Mu + N(u). \tag{26}$$

The vector-function $u \in C^\infty((-\tau, \tau), \mathfrak{U})$ is said to be a *solution to equation (26)*, if for some $\tau \in \mathbb{R}_+$ the function satisfies this equation. The solution $u = u(t)$ to equation (26) is called a *solution to Cauchy problem*

$$u(0) = u_0 \tag{27}$$

for equation (26), if (27) holds for some $u_0 \in \mathfrak{U}$.

Definition 2. [18] *The set $\mathfrak{P} \subset \mathfrak{U}$ is said to be the phase space of equation* (26), *if*

(*i*) *any solution $u = u(t)$ to equation* (26) *belongs to $\mathfrak{P}$ as a trajectory, i.e. $u = u(t) \subset \mathfrak{P}$, $t \in (-\tau, \tau)$;*

(*ii*) *for any $u_0 \in \mathfrak{P}$, there exists a unique solution to problem* (26), (27).

If there exists the operator $L^{-1} \in \mathcal{L}(\mathfrak{U}, \mathfrak{F})$, then (26) is trivially reduced to the equivalent equation

$$\dot{u} = F(u), \tag{28}$$

where the operator $F = L^{-1}(M + N) \in C^\infty(\mathfrak{U})$. Local solvability of problem (27), (28) and, therefore, problem (26), (27) for any $u_0 \in \mathfrak{U}$ is the classical Cauchy theorem. Hence, in this case, the phase space of equation (26) is the whole space $\mathfrak{U}$.

Let $\ker L \neq \{0\}$ and the operator M be $(L, 0)$–bounded, then (26) is reduced to the pair of equivalent equations

$$0 = (\mathbb{I} - Q)(Mu + N(u)), \tag{29}$$

$$\dot{u}^1 = Su^1 + QN(u^0 + u^1), \tag{30}$$

where $u^k \in \mathfrak{U}^k, k = 0, 1$. Consider the set $\mathfrak{M} = \{u \in \mathfrak{U} : (\mathbb{I} - Q)(Mu + N(u)) = 0\}$ that is a candidate for the role of the phase space of equation (26) in this case.

Theorem 8. [18] *Suppose that* $\ker L \neq \{0\}$*, the operator M is $(L, 0)$–bounded, there exists $u_0 \in \mathfrak{M}$, and*

$$\mathbb{I} + M_0^{-1}(\mathbb{I} - Q)N_0' : \mathfrak{U}^0 \to \mathfrak{U}^0 \tag{31}$$

is a topline isomorphism. Then some neighborhood $\mathfrak{O} \subset \mathfrak{M}$ of the point u_0 is a Banach C^∞–manifold modeled by a subspace $\mathfrak{U}^1$, and also belongs to the phase space of equation (26).

3.2. Semilinear Hoff Equation

Let $\mathfrak{U} = \overset{n}{\underset{k=0}{\oplus}}((\mathfrak{H}^1_{k\Delta})^\perp)^{-1}, \mathfrak{F} = \overset{n}{\underset{k=0}{\oplus}}(\mathfrak{H}^1_{k\Delta})^\perp$, where the direct sums are assumed to be "orthogonal", and the space $((\mathfrak{H}^1_{k\Delta})^\perp)^{-1}$ is formally dual to $(\mathfrak{H}^1_{k\Delta})^\perp$. Introduce the operator

$\mathbb{I} = \text{diag}\{\mathbb{I}_k\}$, where $\mathbb{I}_k : (\mathfrak{H}^1_{k\Delta})^\perp \to ((\mathfrak{H}^1_{k\Delta})^\perp)^{-1}$, $k = 0, 1, ..., n$, are the operators of embedding.

For the Hoff equation $(\lambda - \Delta)u_t = \alpha u + \beta u^3$, define the operators

$$L = (\lambda - \Delta)\mathbb{I}, M = \alpha\mathbb{I}, \tag{32}$$

where Δ is the Laplace-Beltrami operator, and $L, M \in \mathcal{L}(\mathfrak{U}, \mathfrak{F})$, while the formulas

$$N = \text{diag}\{N_k\}, (N_k(\xi), \eta)_0 = \beta \int\limits_{\Omega_n} \xi^3 \wedge *\eta,\ \xi, \eta \in (\mathfrak{H}^1_{k\Delta})^\perp \tag{33}$$

define the operator N. Here ξ^3 is a k–form ξ, all coefficients of which are cubed. As a result, we obtain

$$L\dot{u} = Mu + N(u). \tag{34}$$

Lemma 4. [18]
(i) For any $\lambda, \alpha \in \mathbb{R} \setminus \{0\}$, the operator M is (L, p)–bounded with $p = 0$.
(ii) For any $n = 1, 2, ..., 4$, $\beta \in \mathbb{R}$, the operator $N \in C^\infty(\mathfrak{U}, \mathfrak{F})$.

Suppose that $\sigma(\Delta)$ are the eigenvalues of the Laplace–Beltrami operator Δ (see Subsection 1.2), while $\{\varphi_i\}$ is the corresponding set of eigenfunctions on $\mathfrak{U}$.

Let us introduce into consideration the sets

$$\mathfrak{P} = \begin{cases} \mathfrak{U}, \lambda \notin \sigma(\Delta); \\ \{u \in \mathfrak{U} : (u, \varphi_j)_0 = 0, \quad \lambda = \lambda_j\} \end{cases}$$

and

$$\mathfrak{M} = \begin{cases} \mathfrak{U}, \lambda \notin \sigma(\Delta); \\ \{u \in \mathfrak{U} : \alpha(u, \varphi_j)_0 + \beta(N(u), \varphi_j)_0 = 0, \quad \lambda = \lambda_j\}. \end{cases}$$

Theorem 9. [18] *For any $n = \overline{1, 4}$, $\alpha \in \mathbb{R} \setminus \{0\}$, $\beta, \lambda \in \mathbb{R}$, the phase space of equation* (34) *is a simple Banach manifold $\mathfrak{M}$ modeled by the subspace $\mathfrak{P}$.*

Conclusion

In addition to the studies presented in the review, the author wrote a number of papers on the solvability of the equations in spaces of differential forms with stochastic coefficients [10, 11, 16, 17] based on studies for stochastic equations [2, 3, 25]. Results on the stability of solutions in spaces of differential forms with stochastic coefficients were obtained in [6–11, 17]. Also, the studies [13–15] on numerical solutions to these equations were published by the author. In addition to these studies, there exist other areas of study of Sobolev-type equations: high-order Sobolev-type equations [34]; equations on graphs [27]; in areas of optimal control and measurement [12, 19, 23]; multipoint initial-final value problems [4]; complex physical models [20, 21]. These areas are of interest for future study in spaces of differential forms.

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УРАВНЕНИЯ СОБОЛЕВСКОГО ТИПА В ПРОСТРАНСТВАХ ДИФФЕРЕНЦИАЛЬНЫХ ФОРМ НА РИМАНОВЫХ МНОГООБРАЗИЯХ БЕЗ КРАЯ

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Статья содержит обзор результатов, полученных автором как самостоятельно, так и в соавторстве с другими представителями Челябинской научной школы Г.А. Свиридюка по «Уравнениям соболевского типа» в специфических пространствах, а именно пространствах дифференциальных форм, заданных на каком-либо римановом многообразии без края. Уравнения соболевского типа относятся к неклассическим уравнениям математической физики и характеризуются необратимым оператором при старшей производной. При рассмотрении в наших пространствах пришлось использовать специальные обобщения операторов на пространство дифференциальных форм, в частности, оператор Лапласа заменили на его обобщение – оператор Лапласа – Бельтрами. Рассмотрены конкретные интерпритации уравнений с относительно ограниченными операторами: линейное Баренблатта – Желтова – Кочиной, линейное и полулинейное Хоффа, линейное Осколкова. Для этих уравнений исследованы в различных случаях разрешимость задач Коши, Шоуолтера – Сидорова и начально-конечной. В зависимости от выбора типа уравнения (линейное или полулинейное) применялась соответствующая модификация метода фазового пространства. Для использования этого метода, основанного на расщеплении области определения и действия соответствующих операторов, в пространствах дифференциальных форм базой служит теорема Ходжа – Кодаиры о расщеплении области определения оператора Лапласа – Бельтрами.

Ключевые слова: уравнения соболевского типа; метод фазового пространства; дифференциальные формы; риманово многообразие без края.

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