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# ANALYSIS OF THE INFLUENCE OF THE LAGRANGE MULTIPLIER ON THE OPERATION OF THE ALGORITHM FOR ESTIMATING THE SIGNAL PARAMETERS UNDER A PRIORI UNCERTAINTY

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The paper considers a recurrent regularizing algorithm for joint estimation of distortions of a $M$-ary quadrature amplitude modulation ($M$-QAM) signal obtained in a direct conversion receiver path. The algorithm is synthesized using a modified least squares method in the form of Tikhonov's functional under conditions of a priori uncertainty about the laws of noise distribution. The resulting procedure can work both on the test sequence and on information symbols after the detection procedure. We analyze the influence of the Lagrange multiplier on the accuracy of the estimation procedure and on the complexity of the algorithm. It is shown that, with the same accuracy, the regularizing algorithm requires significantly fewer iterations than the procedure without the Lagrange multiplier, and therefore has a lower computational complexity.

*Keywords: regularizing algorithm; a priori uncertainty; modified least squares method; direct transform receiver.*

## Introduction

From the estimation theory, it is well known that the joint estimation of the unknown signal parameters has a higher accuracy than the separate estimation. But the main disadvantage of joint estimation algorithms is their complexity [1, 2].

In communication technology, there exist incorrectly posed problems that require a special approach. This situation arises, for example, when the operator (linear or nonlinear) describing the observed process does not have the opposite one, or is determined with errors leading to the divergence of the computational algorithm used. Such problems are solved by introducing a regularizing parameter into the estimation procedure [3–5].

## 1. Problem Statement

Consider the quadrature components $\mathbf{z}_i = \mathbf{S}_i(\boldsymbol{\Theta}_i)$ of the signal $M$-QAM, $M = 2^{2k}, k \in \mathbb{N}$, observed against the background noise $\boldsymbol{\mu}_i$ with an unknown probability distribution: $\mathbf{y}_i = \mathbf{z}_i + \boldsymbol{\mu}_i$, where $i = 1, 2, ...m$ is a discrete time, $\mathbf{S}_i(\cdot)$ is a nonlinear vector-function describing the quadrature components of the signal:

$$\mathbf{S}_i(\boldsymbol{\Theta}) = \begin{pmatrix} a(I_i \cos(2\pi\Delta f \Delta t i + \varphi_i) - J_i \sin(2\pi\Delta f \Delta t i + \varphi_i)) + b_c \\ \gamma a(I_i \sin(2\pi\Delta f \Delta t i + \varphi_i + \Delta\varphi) + J_i \cos(2\pi\Delta f \Delta t i + \varphi_i + \Delta\varphi)) + b_s \end{pmatrix}. \quad (1)$$

Here $\boldsymbol{\Theta} = (a, \varphi_i, \Delta f, \gamma, \Delta\varphi, b_c, b_s)^T$ is a vector of estimated parameters, $I_i, J_i$ are information amplitudes or symbols of the test sequence taking discrete values, $\varphi_i = \varphi_0 + \alpha_i$ is a random phase formed by the phases of the generators on the transmit-receive side and the delay in the propagation channel, $\varphi_0$ is a phase constant component, $\alpha_i$ is a phase noise, $\Delta f$ is a frequency shift after demodulation, $\Delta t$ is a sampling interval, $a$ is a signal amplitude, $\gamma$, $\Delta\varphi$ are amplitude and phase imbalance, $b_c$, $b_s$ are constant components of the signal quadratures (DC offset).

The problem of finding estimate $\widehat{\mathbf{\Theta}}$ of the vector $\mathbf{\Theta}$ was solved under the following conditions:
– random process $\boldsymbol{\mu}_i$ is stationary, $\mathrm{E}(\boldsymbol{\mu}_i) = \bar{\mathbf{0}}_{2\times1}$, $\mathrm{E}(\boldsymbol{\mu}_i\boldsymbol{\mu}_i^T) = \sigma_\mu^2\mathbf{I}_{2\times2}$ is an observation noise covariance matrix, $\mathrm{E}(\boldsymbol{\mu}_i\boldsymbol{\mu}_j^T) = \mathbf{0}_{2\times2}$ for $i \neq j$, $\mathrm{E}(\,\cdot\,)$ is expectation operator, $\mathbf{I}_{2\times2}$ is identity matrix of the size $2\times2$,
– phase noise $\alpha_i$ is described by a 2nd order moving average model,
– sequence of symbols $I_i$, $J_i$ is known.

## 2. Solution

We form a $2m$ sample vector of the observed process $\mathbf{Y}_m = \begin{pmatrix} \mathbf{y}_m^T & \mathbf{y}_{m-1}^T & \cdots & \mathbf{y}_1^T \end{pmatrix}^T$. Then the observation equation can be written as $\mathbf{Y}_m = \bar{\mathbf{S}}(\mathbf{\Theta}) + \bar{\boldsymbol{\mu}}$, where $\bar{\mathbf{S}}(\mathbf{\Theta}) = \begin{pmatrix} \mathbf{S}_m^T(\mathbf{\Theta}) & \mathbf{S}_{m-1}^T(\mathbf{\Theta}) & \cdots & \mathbf{S}_1^T(\mathbf{\Theta}) \end{pmatrix}^T$, $\bar{\boldsymbol{\mu}} = \begin{pmatrix} \boldsymbol{\mu}_m^T & \boldsymbol{\mu}_{m-1}^T & \cdots & \boldsymbol{\mu}_1^T \end{pmatrix}^T$. The algorithm to find the estimation $\mathbf{\Theta}$ is sought in the class of recurrent procedures. Expanding the nonlinear function $\bar{\mathbf{S}}(\cdot)$ in a Taylor series up to linear term at the point $\widehat{\mathbf{\Theta}}_{l-1}$, where $l$ is the iteration number, we get: $\mathbf{Y}_m \cong \mathbf{D}_{l-1}\mathbf{F}(\mathbf{\Theta}_l) + \bar{\boldsymbol{\mu}}$, where $\mathbf{F}(\mathbf{\Theta}_l) = \mathbf{f}_l = \begin{pmatrix} 1 & \mathbf{\Theta}_l^T \end{pmatrix}^T$, $\mathbf{\Theta}_l = \mathbf{L}\mathbf{f}_l$, $\mathbf{L} = \begin{pmatrix} \mathbf{0}_{7\times1} & \mathbf{I}_{7\times7} \end{pmatrix}$. The model of the estimated vector has the form: $\mathbf{\Theta}_l = \mathbf{\Theta}_{l-1} + \boldsymbol{\xi}_l$, where $\boldsymbol{\xi}_l$ is a shaping noise with zero expectation and the covariance matrix $\mathrm{E}(\boldsymbol{\xi}_l\boldsymbol{\xi}_l^T) = \sigma_\xi^2\mathbf{I}_{7\times7}$, $\sigma_\xi^2 \to 0, \mathbf{I}_{7\times7}$ is identity matrix. We calculate the value of the function $\mathbf{F}(\cdot)$ from the left and right sides of this model and expand it in a Taylor series up to the first approximation at the point $\mathbf{\Theta}_{l-1}$. As a result, we obtain a model linearized with respect to the variable $\mathbf{f}_{l-1}$: $\mathbf{f}_l = \mathbf{f}_{l-1} + \mathbf{W}_{l-1}\boldsymbol{\xi}_l$, where $\mathbf{W}_{l-1} = \mathbf{F}'(\mathbf{\Theta}_{l-1})$ is the first derivative of $\mathbf{F}(\cdot)$ at the point $\mathbf{\Theta}_{l-1}$. Further, the problem of finding the estimate is sought by the modified least squares method in the form of the Tikhonov functional:

$$\Phi(\mathbf{f}_l) = \|\mathbf{Y}_m - \mathbf{D}_{l-1}\mathbf{f}_l\|^2_{\mathbf{Q}^{-1}} + \lambda_l \left\|\mathbf{f}_l - \widehat{\mathbf{f}}_{l-1}\right\|^2_{\mathbf{P}_l^{-1}}, l = 1, 2, ..., M_0. \tag{2}$$

Here $\lambda_l$ is a regularizing Lagrange multiplier, Euclidean norms are determined taking into account the weight matrices $\mathbf{Q} = \sigma_\mu^2\mathbf{I}_{2m\times2m}$ and $\mathbf{P}_l$: $\|\mathbf{Y}_m - \mathbf{D}_{l-1}\mathbf{f}_l\|^2_{\mathbf{Q}^{-1}} = (\mathbf{Y}_m - \mathbf{D}_{l-1}\mathbf{f}_l; \mathbf{Q}^{-1}(\mathbf{Y}_m - \mathbf{D}_{l-1}\mathbf{f}_l))$, $\left\|\mathbf{f}_l - \widehat{\mathbf{f}}_{l-1}\right\|^2_{\mathbf{P}_l^{-1}} = (\mathbf{f}_l - \widehat{\mathbf{f}}_{l-1}; \mathbf{P}_l^{-1}(\mathbf{f}_l - \widehat{\mathbf{f}}_{l-1}))$, $(\bullet; \bullet)$ is a scalar product. By minimizing (2) over $\mathbf{f}_l$ under the constraint conditions $\frac{\left\|\mathbf{Y}_m - \bar{\mathbf{S}}(\widehat{\mathbf{\Theta}}_l)\right\|^2}{2(m-1)} = \sigma_\mu^2$, we obtain expressions for the estimates:

$$\widehat{\mathbf{\Theta}}_l = \widehat{\mathbf{\Theta}}_{l-1} + \mathbf{L}\mathbf{K}_l(\mathbf{Y}_m - \bar{\mathbf{S}}(\widehat{\mathbf{\Theta}}_{l-1})), \quad l = 1, 2, ..., M_0, \tag{3}$$

where $\mathbf{K}_l = (\sigma_\mu^2\mathbf{I} + \lambda_l\mathbf{P}_l\mathbf{D}_{l-1}^T\mathbf{D}_{l-1})^{-1}\lambda_l\mathbf{P}_l\mathbf{D}_{l-1}^T$, $\mathbf{P}_l = \mathbf{\Gamma}_{l-1} + \mathbf{W}\mathbf{W}^T\sigma_\zeta^2$, $\mathbf{\Gamma}_l = (\mathbf{I} - \mathbf{K}_l\mathbf{D}_{l-1})\mathbf{P}_l(\mathbf{I} - \mathbf{K}_l\mathbf{D}_{l-1})^T + \mathbf{K}_l\mathbf{Q}\mathbf{K}_l^T + (\mathbf{I} - \mathbf{K}_l\mathbf{D}_{l-1})\mathbf{P}_{l-1,f\mu}\mathbf{K}_l^T + \mathbf{K}_l\mathbf{P}_{l-1,f\mu}^T(\mathbf{I} - \mathbf{K}_l\mathbf{D}_{l-1})^T$, $\mathbf{P}_{l-1,f\mu} = E(\hat{\mathbf{f}}_{l-1}\boldsymbol{\mu}^T) = \mathbf{K}_{l-1}\mathbf{Q} + (\mathbf{I} - \mathbf{K}_{l-1}\mathbf{D}_{l-2})\mathbf{P}_{l-2,f\mu}$, $\mathbf{I}$ is identity matrix of the size $8\times8$, the initial conditions are $\mathbf{P}_{0,f\mu} = \mathbf{0}_{8\times2m}$, $\mathbf{\Gamma}_0$, $\widehat{\mathbf{\Theta}}_0$ are found from a priori information.

The Lagrange multiplier is calculated by the formula:

$$\lambda_l \approx \frac{\sqrt{2m}\,\sigma_\mu\left(\frac{\left\|\mathbf{Y}_m-\bar{\mathbf{S}}(\widehat{\boldsymbol{\Theta}}_{l-1})\right\|}{\sqrt{2(m-1)}}-\sigma_\mu\right)}{\left\|\mathrm{diag}(\mathbf{D}_{l-1}\mathbf{P}_l\mathbf{D}_{l-1}^T)\right\|}. \tag{4}$$

An approximate expression is obtained for the number of arithmetic operations for two variants of algorithm (3) taking into account the calculation of Lagrange multiplier (4): $N_{(3)(4)} \cong (1018m + 1588)M_0$. If the iterative update of the Lagrange multiplier value is not performed, i.e. $\lambda_l \equiv 1$, then the number of arithmetic operations can be estimated in the form $N_{(3)} \cong (984m + 1585)M_0$.

## 3. Computational Experiment

A computational experiment was carried out with the following data: $M = 64$, $\gamma = 0,5$, $\Delta f = 180.7$ Hz, $\Delta t = 0,25$ $\mu$s, $\varphi_0 = \pi/12$, $\Delta\varphi = \pi/18$, $b_c = 1,3, b_s = 2$, $a = 3$, $m = 500$, detection was carried out using an information sequence of 2000 symbols, the phase noise is stationary, the standard deviation of the phase noise is one degree, the additive noise is Gaussian, the number of realizations is 100. Initial values are taken as follows: $\widehat{\boldsymbol{\Theta}}_0 = (1,0,0,1,0,0,0)^T$, $\boldsymbol{\Gamma}_0 = \begin{pmatrix} 1 & \mathbf{0}_{1\times7} \\ \mathbf{0}_{7\times1} & \sigma_\mu^2\mathbf{I}_{7\times7} \end{pmatrix}$.

Figures 1, 2 show the dependence of the RMSE (root mean square error) of the estimation of some parameters on the number of iterations $l$ for procedure (3) without the Lagrange multiplier ($\lambda_l \equiv 1$) and regularizing algorithm (3), (4) with ratio of signal to noise equal to 27 dB for in-phase component.

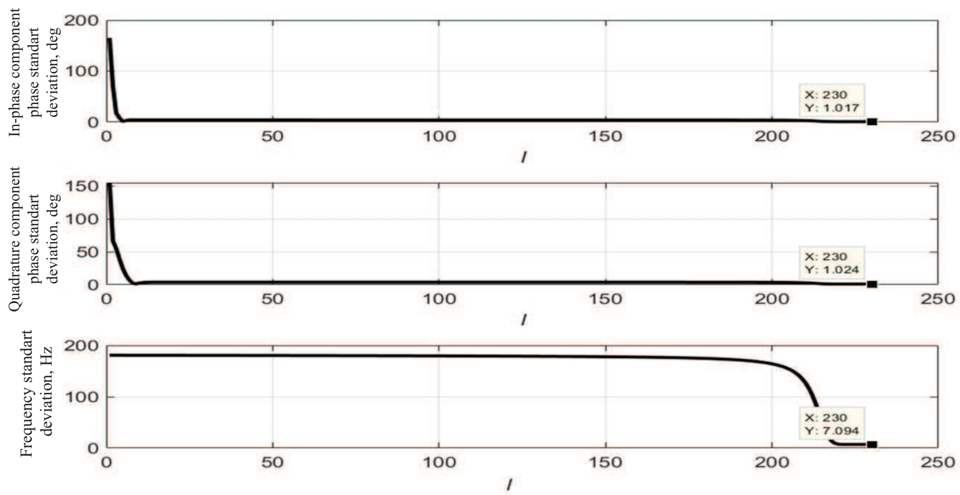



**Fig. 1**. Dependence of the standard deviation of the estimation of the frequency and the total phase of the 64-QAM signal on the number of iterations for algorithm (3) without Lagrange multiplier

The above figures show that the algorithm with the Lagrange multiplier has a significantly shorter transient process. Therefore, it has less computational complexity. Regularizing algorithm (3), (4) converges already after 10 iterations. Algorithm (3) with

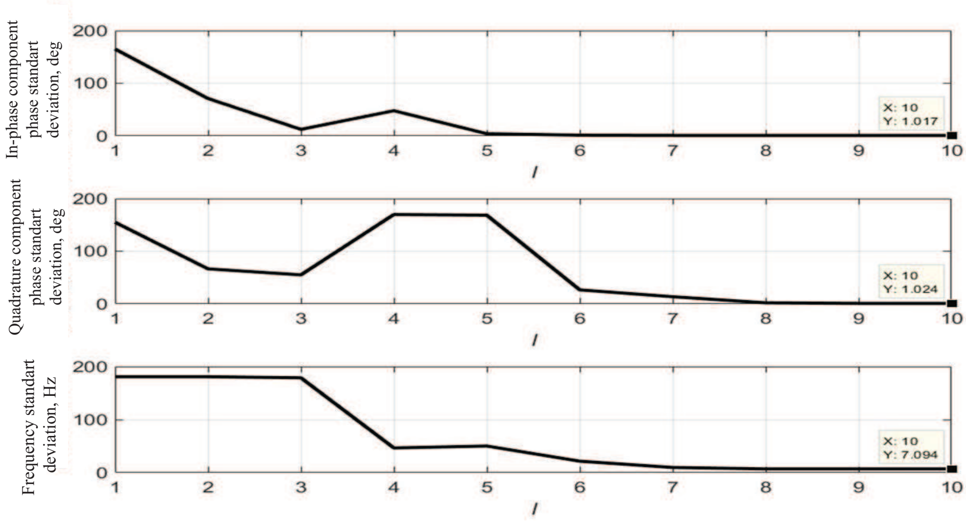
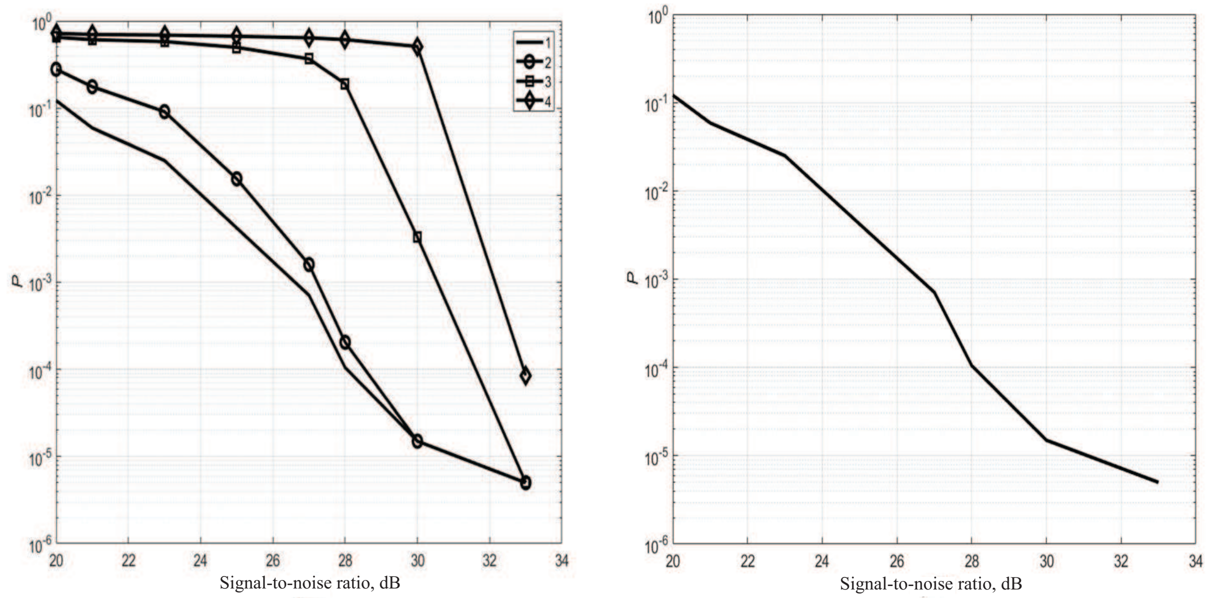



**Fig. 2**. Dependence of the standard deviation of the estimation of the frequency and the total phase of the 64-QAM signal on the number of iterations for algorithm (3) with Lagrange multiplier




a) b)

**Fig. 3**. Experimental curves of noise immunity of 64-QAM signal reception during algorithm (3) operation: without Lagrange multiplier ($\lambda_l \equiv 1$ ) and with a different number of iterations $M_0$(**1** – $M_0 = 230$**;** **2** – $M_0 = 225$**;** **3** – $M_0 = 220$**;** **4** –$M_0 = 215$) – **a)**, with Lagrange multiplier (4), $M_0 = 10$ – **b)**

$\lambda_l \equiv 1$, which is a recurrent least squares (RLS) method with weighted matrices, converges only after $M_0 = 230$. Then, with the size of the estimated vector equal to 7 and the length of the test sequence equal to 500, the complexity of the RLS method exceeds the complexity of regularizing algorithm (3), (4) by approximately $\frac{N_{(3)}}{N_{(3)(4)}} \cong 22$ times. Figure 3 illustrates the experimental probabilities of error per symbol for receiving a 64-QAM signal when using algorithm (3) with $\lambda_l \equiv 1$ and a different number of iteration steps $M_0$, and regularizing algorithm (3), (4). Figure 3 shows that the noise immunity, which regularizing algorithm (3), (4) allows for 10 iterations, is achieved using procedure (3) without the Lagrange multiplier only at 230 iterations.

A computational experiment showed that to achieve the same noise immunity, regularizing algorithm (3), (4) requires fewer iterations, and hence fewer arithmetic operations, than recurrent least squares method (3) ($\lambda_l \equiv 1$). With the length of the estimated vector equal to 7 and the length of the test sequence equal to 500 symbols, the complexity of the recurrent least squares method exceeds the complexity of regularizing algorithm (3), (4) by 22 times.

## References

1. Hsu C.-J., Cheng R., Sheen W.-H. Sheen Joint Least Squares Estimation of Frequency, DC Offset, I-Q Imbalance, and Channel in MIMO Receivers. *IEEE Transactions on Vehicular Technology*, 2009, vol. 58, no. 5, pp. 2201–2213. DOI: 10.1109/TVT.2008.2005989
2. Weikert O. Joint Estimation of Carrier and Sampling Frequency Offset, Phase Noise, IQ Offset and MIMO Channel for LTE Advanced UL MIMO. *IEEE 14th Workshop on Signal Processing Advances in Wireless Communications*, 2013, Darmstadt, pp. 520–524. DOI: 10.1109/SPAWC.2013.6612104.
3. Tikhonov A.N., Leonov A.S., Yagola A.G. *Nelineynye nekorrektnye zadachi* [Nonlinear Incorrect Tasks]. Moskva, Nauka Fizmatlit, 1995. (in Russian)
4. Bakushinskiy A.B., Kokurin M.Yu. [Iterative Stochastic Approximation Methods for Solving Irregular Nonlinear Operator Equations]. *Computational Mathematics and Mathematical Physics*, 2015, vol. 55, no. 10, pp. 1637–1645. (in Russian)
5. Golubev G.K. [Concentrations of Risks of Convex Combinations of Linear Estimates]. *Problemy peredachi informatsii*, 2016, vol. 52, no. 4, pp.31–48. (in Russian)
6. Poborchaya N.E. [Methods for Estimating the Parameters of a Random Signal in Conditions of a Priori Uncertainty]. *Elektrosvyaz'*, 2010, no. 3, pp. 24–26. (in Russian)
7. Poborchaya N.E. Stationary Channel Factors and Signal Disturbances in a Direct Converter Receiver in a System with MIMO Joint Estimation Algorithm. *Systems of Signal Synchronization, Generating and Processing in Telecommunications*, Svetlogorsk, 2020. DOI: 10.1109/SYNCHROINFO49631.2020.9166068.

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## АНАЛИЗ ВЛИЯНИЯ МНОЖИТЕЛЯ ЛАГРАНЖА НА РАБОТУ АЛГОРИТМА ОЦЕНИВАНИЯ ПАРАМЕТРОВ СИГНАЛА В УСЛОВИЯХ АПРИОРНОЙ НЕОПРЕДЕЛЕННОСТИ

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В работе рассматривается рекуррентный регуляризующий алгоритм совместной оценки искажений сигнала многопозиционной квадратурной модуляции (M-QAM), полученных в тракте приемника прямого преобразования. Алгоритм синтезирован с помощью модифицированного метода наименьших квадратов в виде функционала Тихонова в условиях априорной неопределенности относительно законов распределения шумов. Полученная процедура может работать как по тестовой последовательности, так и по информационным символам после процедуры детектирования. Проанализировано влияние множителя Лагранжа на точность процедуры оценивания и на сложность алгоритма. Показано, что при одинаковой точности регуляризующий алгоритм требует существенно меньшее количество итераций, чем процедура без множителя Лагранжа, а значит обладает более низкой вычислительной сложностью.

*Ключевые слова: регуляризующий алгоритм; априорная неопределенность; модифицированный метод наименьших квадратов; приемник прямого преобразования.*

## Литература

1. Hsu, C.-J. Joint Least Squares Estimation of Frequency, DC Offset, I-Q Imbalance, and Channel in MIMO Receivers / C.-J. Hsu, R. Cheng, W.-H. Sheen // IEEE Transactions on Vehicular Technology. – 2009 – V. 58, № 5. – P. 2201–2213.
2. Weikert, O. Joint Estimation of Carrier and Sampling Frequency Offset, Phase Noise, IQ Offset and MIMO Channel for LTE Advanced UL MIMO / O. Weikert // 2013 IEEE 14th Workshop on Signal Processing Advances in Wireless Communications, Darmstadt, 2013. – P. 520–524.
3. Тихонов, А.Н. Нелинейные некорректные задачи / А.Н. Тихонов, А.С. Леонов, А.Г. Ягола. – М.: Наука Физматлит, 1995.
4. Бакушинский, А.Б. Итерационные методы стохастической аппроксимации для решения нерегулярных нелинейных операторных уравнений / А.Б. Бакушинский, М.Ю. Кокурин // Журнал вычислительной математики и математической физики. – 2015. – Т. 55, № 10. – С. 1637–1645.
5. Голубев, Г.К. Концентрации рисков выпуклых комбинаций линейных оценок / Г.К. Голубев // Проблемы передачи информации. – 2016. – Т. 52, № 4. – С. 31–48.
6. Поборчая, Н.Е. Методы совместной оценки дрейфа постоянных составляющих и амплитудно-фазового разбаланса КАМ сигнала на фоне аддитивного белого шума // Электросвязь. – 2013. – № 5. – С.24–26.
7. Poborchaya, N.E. Stationary Channel Factors and Signal Disturbances in a Direct Converter Receiver in a System with MIMO Joint Estimation Algorithm / N.E. Poborchaya // 2020 Systems of Signal Synchronization, Generating and Processing in Telecommunications, Svetlogorsk, 2020.

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