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EXACT SOLUTIONS OF THE (2+1)-DIMENSIONAL KUNDU–MUKHERJEE–NASKAR MODEL VIA IBSEFM

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The aim of this study is to construct the exact solutions of the (2+1)-dimensional Kundu–Mukherjee–Naskar (KMN) equation via Improved Bernoulli Sub-Equation Function Method (IBSEFM). The physics of this model describes optical dromions in (2+1)-dimensional case. It is also studied in fluid dynamics. Applying the proposed method, we obtain new exact solutions of (2+1)-dimensional KMN equation. Moreover, we plot the 2D-3D figures and contour surfaces according to the suitable parameters by the aid of computer software. The results confirm that IBSEFM is powerful, effective and straightforward for solving nonlinear partial differential equations arising in mathematical physics.

Keywords: IBSEFM; Kundu–Mukherjee–Naskar equation; exact solutions.

Introduction

Various natural phenomena in physics are modelled by nonlinear partial differential equations. Investigating the exact solutions of these equations is very important to describe the nonlinear physical process ranging from graviation to fluid dynamics. Models of real-life problems such as nonlinear waves [1], temporal transitions [2], shallow water waves [3] and plasma physics [4] are based on nonlinear phenomena. For a better understanding of these equations, many powerful methods were introduced by mathematicians for exact solutions of nonlinear partial differential equations. Some of them are the Adomian decomposition [5], (G'/G)-expansion [6], the improved tanh function [7], IBSEFM [8], extended direct algebraic method [9] and others.

In recent years, a deep research attention was paid to rogue wave solutions. Recently, in [10], a new completely integrable (2+1)-dimensional nonlinear evolution equation was derived to describe the dynamics of two-dimensional oceanic rogue wave phenomena. In the paper [10], this nonlinear dynamical equation was called the Kundu-Mukherjee-Naskar (KMN) equation. Firstly, the rogue waves were observed in the deep oceans and then derived in modelling nonlinear ion acoustic waves in magnetized plasma system [11]. In literature, there exist only four methods to find exact solutions of (2 + 1)-dimensional KMN equation. These methods are trial equation method [12], extended trial function method [13], traveling wave reduction [14], extended direct algebraic method [15].

In this study, we consider the following (2+1)-dimensional KMN equation:

$$iu_t + \alpha_6 u_{xy} + i\alpha_7 u(uu_x^* - u^* u_x) = 0, \tag{1}$$

where $i = \sqrt{-1}$, $u(x, y, t)$ represents the nonlinear wave function with the independent variables x, y (space coordinates) and t (time), u^* denotes complex conjugation of u. Here, the coefficient α_6 is the dispersion term and the coefficient α_7 is the nonlinearity term. Before beginning of the solution procedure, we should describe the proposed method.

1. Description of IBSEFM

In this section, we give the fundamental properties of the IBSEFM [8]. Let us present the five main steps of the IBSEFM [16–20].

Step 1. Consider the following nonlinear partial differential equation in two independent variables x and t:

$$P(u, u_t, u_{xx}, ...) = 0, \tag{2}$$

where P is the function containing $u, u_t, u_{xx}, ...$ and the subscripts denote the partial derivatives of $u(x,t)$ with respect to x and t. The aim is to convert (2) into the ordinary differential equation with a suitable wave transformation as

$$u(x,t) = V(\xi), \quad \xi = x - \mu t. \tag{3}$$

Using (3), equation (2) turns into the ordinary differential equation of the form

$$N(V, V', V'', ...) = 0, \tag{4}$$

where N is the function of V and its derivatives $V', V'', ...$ with respect to ξ. If we integrate (4) term by term, we obtain integration constants which can be determined later.

Step 2. Hypothesize that the solution of (4) can be presented as follows:

$$V(\xi) = \frac{\sum\limits_{i=0}^{n} a_i F^i(\xi)}{\sum\limits_{j=0}^{m} b_j F^j(\xi)} = \frac{a_0 + a_1 F(\xi) + a_2 F^2(\xi) + ... + a_n F^n(\xi)}{b_0 + b_1 F(\xi) + b_2 F^2(\xi) + ... + b_m F^m(\xi)}, \tag{5}$$

where $a_0, a_1, ..., a_n$ and $b_0, b_1, ..., b_m$ are coefficients to be determined later, $m \neq 0, n \neq 0$ are chosen arbitrary constants of the balance principle, and the form of Bernoulli differential equation is as follows:

$$F'(\xi) = \sigma F(\xi) + dF^M(\xi), \ d \neq 0, \sigma \neq 0, \ M \in \mathbb{R}/\left\{0,1,2\right\}, \tag{6}$$

where $F(\xi)$ is a polynomial.

Step 3. The positive integers m, n, M are found by balance principle that is both nonlinear term and the highest order derivative term of (4).

Substitute (5), (6) in (2) and obtain the equation of polynomial $\Theta(F)$ of F as follows:

$$\Theta(F(\xi)) = \rho_s F(\xi)^s + ... + \rho_1 F(\xi) + \rho_0 = 0,$$

where ρ_i are coefficients to be determined later.

Step 4. The coefficients of $\Theta(F(\xi))$, which give us a system of algebraic equations, turned out to be zero:

$$\rho_i = 0, i = 0, .., s.$$

Step 5. When we solve (4), we get the following two cases with respect to σ and d:

$$F(\xi) = \left[\frac{-de^{\sigma(\epsilon-1)} + \epsilon\sigma}{\sigma e^{\sigma(\epsilon-1)\xi}}\right]^{\frac{1}{1-\epsilon}}, \quad d \neq \sigma, \tag{7}$$

$$F(\xi) = \left[\frac{(\epsilon - 1) + (\epsilon + 1)\tanh(\sigma(1-\epsilon)\frac{\xi}{2})}{1 - \tanh\left(\sigma(1-\epsilon)\frac{\xi}{2}\right)}\right], d = \sigma,\ \epsilon \in \mathbb{R}. \tag{8}$$

Using a complete discrimination system for the polynomial of $F(\xi)$, we obtain the analytical solutions of (4) via mathematics software and categorize the exact solutions of (4). To achieve better results, we can plot two and three dimensional figures of exact solutions by considering proper values of parameters.

2. Application of IBSEFM

In this section, the application of the IBSEFM to the KMN model is presented. Let us consider the following wave transform:

$$u\,(x, y, t) = U\,(\xi)\, e^{\phi(x,y,t)}, \tag{9}$$

where $U(\xi)$ represents the amplitude portion and

$$\xi = \alpha_1 x + \alpha_2 y - \eta t, \tag{10}$$

and the phase portion of the solution is defined as

$$\phi(x, y, t) = -\alpha_3 x - \alpha_4 y + \alpha_5 t. \tag{11}$$

Then, we get the nonlinear ordinary differential equation

$$\alpha_1\alpha_2\alpha_6 U'' - (\alpha_5 + \alpha_3\alpha_4\alpha_6)U - 2\alpha_3\alpha_7 U^3 = 0, \tag{12}$$

$$\eta = -\alpha_6(\alpha_2\alpha_3 + \alpha_1\alpha_4). \tag{13}$$

Then we reconsider (12) for the balance principle considering among U'' and U^3 and get the relationship as follows:

$$M = n - m + 1. \tag{14}$$

Formula (14) shows different cases of the solutions of (1) and we can obtain some exact solutions. According to the balance, we consider $M = 3, m = 1, n = 3$ for (12) and (14), and the following equations hold:

$$U(\xi) = \frac{a_0 + a_1 G(\xi) + a_2 G^2(\xi) + a_3 G^3(\xi)}{b_0 + b_1 G(\xi)} \equiv \frac{\Upsilon(\xi)}{\Psi(\xi)}, \tag{15}$$

$$U'(\xi) = \frac{\Upsilon'(\xi)\Psi\,(\xi) - \Upsilon(\xi)\Psi'(\xi)}{\Psi^2(\xi)}, \tag{16}$$

and

$$U''(\xi) = \frac{\Upsilon'(\xi)\Psi\,(\xi) - \Upsilon(\xi)\Psi'(\xi)}{\Psi^2(\xi)} - \frac{\left[\Upsilon(\xi)\Psi'\,(\xi)\right]'\Psi^2(\xi) - 2\Upsilon(\xi)[\Psi'(\xi)]^2\Psi(\xi)}{\Psi^4(\xi)}, \tag{17}$$

where $G' = \sigma G + dG^3$, $a_3 \neq 0$, $b_1 \neq 0$, $\sigma \neq 0$, $d \neq 0$. Using (15)-(17) in (12), we get from the coefficients of the polynomial of G the following:

$$Constant: -a_0b_0^2\alpha_5 - a_0b_0^2\alpha_3\alpha_4\alpha_6 - 2a_0^3\alpha_3\alpha_7 = 0,$$

$$G: -a_1b_0^2\alpha_5 - 2a_0b_0b_1\alpha_5 - a_1b_0^2\alpha_3\alpha_4\alpha_6 - 2a_0b_0b_1\alpha_3\alpha_4\alpha_6 + \sigma^2a_1b_0^2\alpha_1\alpha_2\alpha_7 - \\ -6a_0^2a_1\alpha_3\alpha_7 = 0,$$

$$G^2: a_2b_0^2\alpha_5 - 2a_1b_0b_1\alpha_5 - a_0b_0^2\alpha_3\alpha_4\alpha_6 - a_2b_0^2\alpha_3\alpha_4\alpha_6 - 2a_1b_0b_1\alpha_3\alpha_4\alpha_6 - \\ -a_0b_1^2\alpha_3\alpha_4\alpha_6 + 4\sigma^2a_2b_0^2\alpha_1\alpha_2\alpha_7 + \sigma^2a_0b_1^2\alpha_1\alpha_2\alpha_7 - 6a_0a_1^2\alpha_3\alpha_7 - 6a_0^2a_2\alpha_3\alpha_7 = 0$$

$$G^3: -a_3b_0^2\alpha_5 - 2a_2b_0b_1\alpha_5 - a_1b_1^2\alpha_5 - a_3b_0^2\alpha_3\alpha_4\alpha_6 - 4d\sigma a_0b_0b_1\alpha_1\alpha_2\alpha_7 - \\ -a_1b_1^2\alpha_3\alpha_4\alpha_6 + 4d\sigma a_1b_0^2\alpha_1\alpha_2\alpha_7 - 2a_2b_0b_1\alpha_3\alpha_4\alpha_6 + 9\sigma^2a_3b_0^2\alpha_1\alpha_2\alpha_7 + \\ +3\sigma^2a_2b_0b_1\alpha_1\alpha_2\alpha_7 - 2a_1^3\alpha_3\alpha_7 - 12a_0a_1a_2\alpha_3\alpha_7 - 6a_0^2a_3\alpha_3\alpha_7 = 0,$$

$$G^4: -2a_3b_0b_1\alpha_5 - a_2b_1^2\alpha_5 - 2a_3b_0b_1\alpha_3\alpha_4\alpha_6 - a_2b_1^2\alpha_3\alpha_4\alpha_6 + 12d\sigma a_2b_0^2\alpha_1\alpha_2\alpha_7 + \\ +11\sigma^2a_3b_0b1\alpha_1\alpha_2\alpha_7 + \sigma^2a_2b_1^2\alpha_1\alpha_2\alpha_7 - 6a_1^2a_2\alpha_3\alpha_7 - 12a_0a_2a_3\alpha_3\alpha_7 = 0,$$

$$G^5: -a_3b_1^2\alpha_5 - a_3b_1^2\alpha_3\alpha_4\alpha_6 + 3d^2a_1b_0^2\alpha_1\alpha_2\alpha_7 + 24d\sigma a_3b_0^2\alpha_1\alpha_2\alpha_7 - 3d^2a_0b_0b_1\alpha_1\alpha_2\alpha_7 + \\ +12d\sigma a_2b_0b_1\alpha_1\alpha_2\alpha_7 + 4\sigma^2a_3b_1^2\alpha_1\alpha_2\alpha_7 - 6a_1a_2^2\alpha_3\alpha_7 - 6a_1^2a_3\alpha_3\alpha_7 - \\ -12a_0a_2a_3\alpha_3\alpha_7 = 0,$$

$$G^6: 8d^2a_2b_0^2\alpha_1\alpha_2\alpha_7 + d^2a_1b_0b_1\alpha_1\alpha_2\alpha_7 + 32d\sigma a_3b_0b_1\alpha_1\alpha_2\alpha_7 - d^2a_0b_1^2\alpha_1\alpha_2\alpha_7 \\ +4d\sigma a_2b_1^2\alpha_1\alpha_2\alpha_7 - 2a_2^3\alpha_3\alpha_7 - 12a_1a_2a_3\alpha_3\alpha_7 - 6a_0a_3^2\alpha_3\alpha_7 = 0,$$

$$G^7: 15d^2a_3b_0^2\alpha_1\alpha_2\alpha_7 + 9d^2a_2b_0b_1\alpha_1\alpha_2\alpha_7 + 12d\sigma a_3b_1^2\alpha_1\alpha_2\alpha_7 - 6a_2^2a_3\alpha_3\alpha_7 - \\ -6a_1a_3^2\alpha_3\alpha_7 = 0,$$

$$G^8: 21d^2a_3b_0b_1\alpha_1\alpha_2\alpha_7 + 3d^2a_2b_1^2\alpha_1\alpha_2\alpha_7 - 6a_1a_3^2\alpha_3\alpha_7 = 0,$$

$$G^9: 8d^2a_3b_1^2\alpha_1\alpha_2\alpha_7 - 2a_3^3\alpha_3\alpha_7 = 0.$$

By solving the above system, we obtain the coefficients as follows.

Case 1. For $\sigma \neq d$, we can consider the following coefficients obtained by a software:

$$\begin{aligned} a_0 &= -i\frac{\sqrt{\alpha_3}\sqrt{\alpha_4}\sqrt{\alpha_6}b_0}{\sqrt{2}\sqrt{\alpha_3}\sqrt{\alpha_7}}; a_1 = -i\frac{\sqrt{\alpha_3\alpha_4\alpha_6}b_1}{\sqrt{2}\sqrt{\alpha_3}\sqrt{\alpha_7}}; \\ a_2 &= \frac{-i\sqrt{2}d\sqrt{\alpha_3\alpha_4\alpha_6+\alpha_5}b_0}{\sqrt{\alpha_3}\sqrt{\alpha_7}\sigma}; a_3 = \frac{-i\sqrt{2}d\sqrt{\alpha_3\alpha_4\alpha_6+\alpha_5}b_1}{\sqrt{\alpha_3}\sqrt{\alpha_7}\sigma}. \end{aligned}$$

Substitute these coefficients along with (7) in (15) and obtain the complex solution of (1):

$$u_1(x,y,t) = -\frac{ie^{-i(-\alpha_5t+(x+\alpha_4y))}\sqrt{\alpha_3\alpha_4\alpha_6}\left(\frac{2d}{exp\left\{-\frac{(\alpha_3\alpha_4\alpha_6)(\alpha_4\alpha_6t-x)}{\alpha_2\alpha_6\sigma}+2\alpha_2(\alpha_3\alpha_6t-y)\sigma\right\}\epsilon-\frac{d}{\sigma}}+\sigma\right)}{\sqrt{2}\sqrt{\alpha_3}\sqrt{\alpha_7}},$$

where $\alpha_2, \alpha_3, \alpha_6, \alpha_7, \sigma$ are not equal to zero.

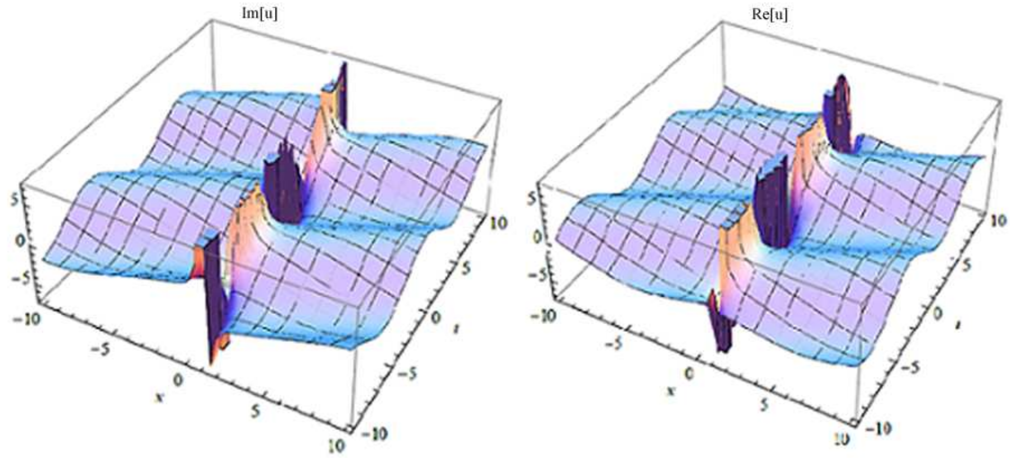

Fig. 1. 3D-plots of $u_1(x,y,t)$ for the values $d = 0,4$; $\sigma = 0,4$; $\alpha_2 = 0,9$; $\alpha_3 = 0,5$; $\alpha_4 = 0,3$; $\alpha_5 = 0,25$; $\alpha_6 = 0,1$; $\alpha_6 = 0,7$; $\alpha_7 = 0,3$; $\epsilon = 0,2$; $-15 < x < 15$, $-15 < t < 15$

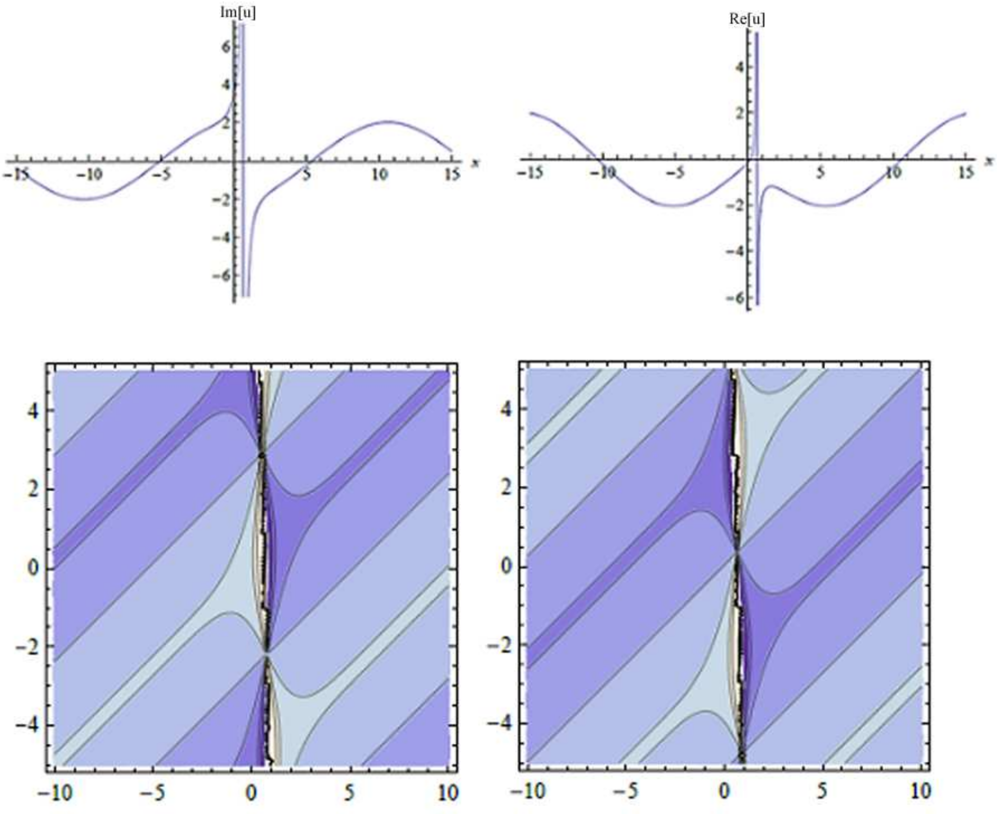

Fig. 2. 2D-graphs of $u_1(x,y,t)$ for $-15 < x < 15$, $t = 0,1$ and contour surfaces for $-10 < x < 10$, $0 < t < 1$

Case 2. For $\sigma \neq d$, we consider the following coefficients obtained by a software:

$$
\begin{aligned}
a_0 &= -\frac{\sqrt{\alpha_1}\sqrt{\alpha_2}\sqrt{\alpha_6}\sigma b_0}{\sqrt{\alpha_3}\sqrt{\alpha_7}}; a_1 = -\frac{\sqrt{\alpha_1}\sqrt{\alpha_2}\sqrt{\alpha_6}\sigma b_1}{\sqrt{\alpha_3}\sqrt{\alpha_7}};\\
a_3 &= -\frac{2d\sqrt{\alpha_1}\sqrt{\alpha_2}\sqrt{\alpha_6}\sigma b_1}{\sqrt{\alpha_3}\sqrt{\alpha_7}}; \quad \alpha_4 = -\frac{\alpha_5 + 2\alpha_1\alpha_2\alpha_6\sigma^2}{\alpha_3\alpha_6}.
\end{aligned}
$$

Substitute these coefficients along with (7) in (15) and obtain the complex solution of (1):

$$
\begin{aligned}
u_2(x,y,t) &= -\frac{\left(\dfrac{2d}{exp\left\{-\frac{2\sigma\left(\alpha_1\alpha_5-\alpha_2\alpha_3^2\alpha_6+2\alpha_1^2\alpha_2\alpha_6\sigma^2\right)t+\alpha_1\alpha_3x+\alpha_2\alpha_3y}{\alpha_3}\right\}\epsilon-\frac{d}{\sigma}}+\sigma\right)}{\sqrt{-\alpha_1\alpha_2\alpha_6}+\sqrt{\alpha_3}\sqrt{\alpha_7}}\times\\
&\times \quad \frac{ie^{i(\alpha_5 t-\alpha_3 x)}\sqrt{-\alpha_1\alpha_2\alpha_6}}{\sqrt{-\alpha_1\alpha_2\alpha_6}+\sqrt{\alpha_3}\sqrt{\alpha_7}},
\end{aligned}
$$

where $\alpha_1, \alpha_2, \alpha_3, \alpha_6, \alpha_7, \sigma$ are constants different from zero.

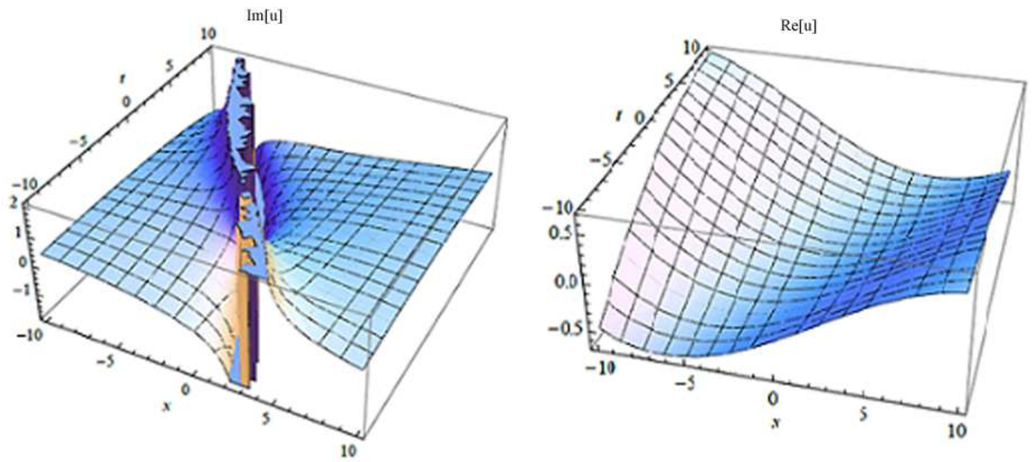

Fig. 3. 3D-plots of $u_2(x,y,t)$ for the values $d = 0,5$; $\sigma = 0,4$; $\alpha_3 = 0,2$; $\alpha_4 = 0,8$; $\alpha_5 = 0,1$; $\alpha_6 = 0,7$; $\epsilon = 0,1$; $-10 < x < 10$, $-10 < t < 10$

Case 3. For $\sigma \neq d$,

$$
\begin{aligned}
a_0 &= -\frac{\sqrt{-\alpha_3\alpha_4\alpha_6-\alpha_5}a_2}{2\sqrt{2}d\sqrt{\alpha_1}\sqrt{\alpha_2}\sqrt{\alpha_6}}; \quad a_1 = \frac{-\sqrt{-\alpha_3\alpha_4\alpha_6-\alpha_5}a_2b_1}{2\sqrt{2}d\sqrt{\alpha_1}\sqrt{\alpha_2}\sqrt{\alpha_6}b_0}; \quad a_3 = \frac{a_2b_1}{b_0};\\
\alpha_7 &= \frac{4d^2\alpha_1\alpha_2\alpha_6 b_0^2}{\alpha_3 a_2^2}; \quad \sigma = -\frac{\sqrt{-\alpha_3\alpha_4\alpha_6-\alpha_5}}{\sqrt{2}\sqrt{\alpha_1}\sqrt{\alpha_2}\sqrt{\alpha_6}}.
\end{aligned}
$$

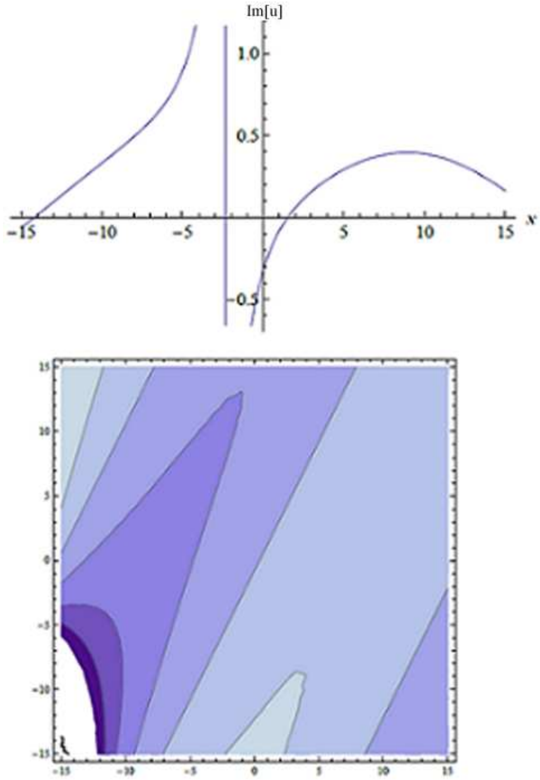

Fig. 4. 2D-graphs of $u_2(x,y,t)$ for $-15 < x < 15$, $t = 0,2$ and contour surfaces for $-15 < x < 15$, $-15 < t < 15$

Substitute these coefficients along with (7) in (15) and obtain the complex exponential function solution as follows:

$$
\begin{aligned}
u_3(x,y,t) \;&=\; -\frac{\left(\dfrac{2d}{exp\left\{-\frac{2\sigma\left(\alpha_1\alpha_5-\alpha_2\alpha_3^2\alpha_6+2\alpha_1^2\alpha_2\alpha_6\sigma^2\right)t+\alpha_1\alpha_3x+\alpha_2\alpha_3y}{\alpha_3}\right\}\epsilon-\frac{d}{\sigma}}+\sigma\right)}{2\sqrt{\alpha_3}\sqrt{-\alpha_3\alpha_4\alpha_6}-\alpha_5\sqrt{\frac{d^2\alpha_1\alpha_2\alpha_6+b_0^2}{\alpha_3a_2^2}}}\times \\
&\times\;\frac{ie^{-i(\alpha_5t+\alpha_3x+\alpha_4y)}\sqrt{\alpha_1}\sqrt{\alpha_2}\sqrt{\alpha_6}\sqrt{\alpha_3\alpha_4\alpha_6+\alpha_5}}{2\sqrt{\alpha_3}\sqrt{-\alpha_3\alpha_4\alpha_6}-\alpha_5\sqrt{\frac{d^2\alpha_1\alpha_2\alpha_6+b_0^2}{\alpha_3a_2^2}}},
\end{aligned}
$$

where $\alpha_1,\alpha_2,\alpha_3,\alpha_6,\alpha_7,\sigma\neq 0$.

Conclusion

In this work, the Improved Bernoulli Sub-Equation Function Method (IBSEFM) is used in (2+1)-dimensional Kundu–Mukherjee–Naskar (KMN) equation. Various solitary wave solutions are successfully constructed. Using mathematics software, two and three dimensional figures and contourplot surfaces of the solutions are plotted according to the suitable values of parameters. According to the results, we see that the formats of travelling

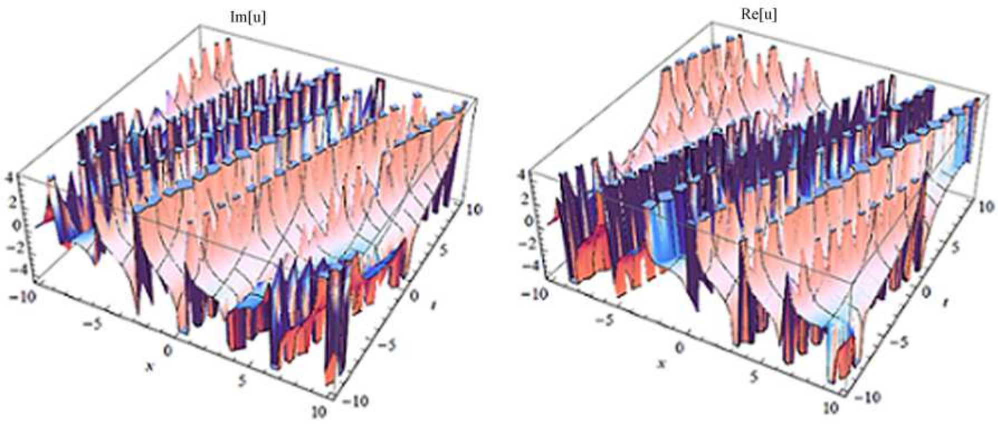

Fig. 5. 3D-plots of $u_3(x, y, t)$ for the values $d = 0,3$; $\sigma = 0,4$; $\alpha_1 = 0,6$; $\alpha_2 = 0,5$; $\alpha_3 = 0,2$; $\alpha_4 = 0,1$; $\alpha_5 = 0,9$; $\alpha_1 = 0,3$; $\alpha_7 = 0,1$; $\epsilon = 0,1$; $-10 < x < 10$, $-10 < t < 10$

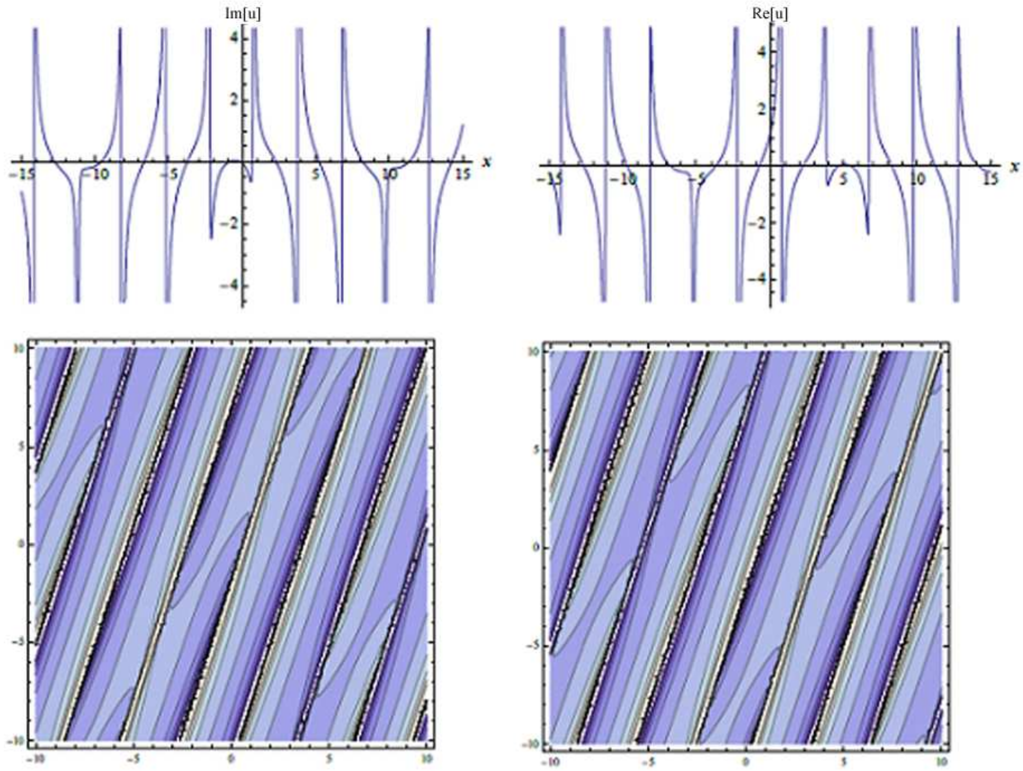

Fig. 6. 2D-graphs of $u_3(x, y, t)$ for $-15 < x < 15$, $t = 0,3$ and contour surfaces for $-10 < x < 10$, $-10 < t < 10$

wave solutions in two and three dimensional surfaces are similar to the physical meaning of results. The solutions are also solitary wave solutions. It is also clear that the more steps are developed and the better approximations are obtained. The conclusions show that the IBSEFM is simple, effective and powerful. Therefore, the IBSEFM is applicable to solve different kind of nonlinear partial differential equations in mathematical physics.

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ТОЧНЫЕ РЕШЕНИЯ (2+1)-МЕРНОЙ МОДЕЛИ КУНДУ – МУХЕРДЖИ – НАСКАРА С ПОМОЩЬЮ IBSEFM

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Целью данного исследования является построение точных решений (2+1)-мерного уравнения Кунду–Мухерджи–Наскара (КМН) с помощью усовершенствованного метода функций подуравнений Бернулли (IBSEFM). Физика этой модели описывает оптические дромионы в (2+1)-мерном пространстве, что также изучается в гидродинамике. Применяя предложенный метод, получаем новые точные решения (2+1)-мерного уравнения КМН. Кроме того, мы наносим 2D-3D фигуры и контурные поверхности в соответствии с подходящими параметрами с помощью компьютерного программного обеспечения. Результаты подтверждают, что IBSEFM является мощным, эффективным и простым средством решения нелинейных дифференциальных уравнений в частных производных, возникающих в математической физике.

Ключевые слова: IBSEFM; уравнение Кунду – Мукерджи – Наскара; точные решения.

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