MSC 78A25, 78A40 DOI: 10.14529/mmp220310

# ALGORITHM FOR VERIFYING THE MEASUREMENTS

***A.L. Shestakov*[1], *D.S. Klygach*[1], *M.G. Vakhitov*[1]**
[1]South Ural State University, Chelyabinsk, Russian Federation
E-mail: a.l.shestakov@susu.ru, klygachds@susu, vmg@kipr.susu.ac.ru

This paper describes the Kramers–Kronig relation for verifying the obtained values of $S$-parameters for different operation conditions of a transmission line. We obtain and prove lemmas for $S$-parameters for operation conditions of the line under short-circuit, open-circuit, and matched load. We give a comparison of theoretical and experimental values, which confirm the correctness of the obtained relations and conclusions.

*Keywords: Kramers–Kronig relation; measurement; electrodynamic parameters; measurement verifying.*

## Introduction

One of the most important problems in the theory of measurements and the determination of physical quantity values is verifying the validity of the results obtained and their compliance with the proposed mathematical or physical model of the measurement process. When measuring the electrodynamic parameters of composite materials, there exists no a reliable algorithm for verifying the obtained measurement results for compliance with the parameters of the materials under study. A number of studies suggests that the Kramers–Kronig relation can be used as such verification [1–4].

The Kramers–Kronig relations integrally connect the real and imaginary parts of any complex function that is analytic in the upper half-plane. The relations are often used to describe the connection between the real and imaginary parts of response functions in physical systems, because the analyticity of the response function implies that the system satisfies the principle of causality, and vice versa [5, 6].

In particular, the Kramers–Kronig relations express the relationship between the real and imaginary parts of the permittivity in classical electrodynamics as well as the probability amplitudes of the transition (matrix element) between two states in quantum field theory. In mathematics, the Kramers–Kronig relations are known as the Hilbert transform [7–9].

## 1. Mathematical Model of Verifying Measurement Results

In mathematics and in signal processing, the Hilbert transform is a linear operator that, for each function $u(t)$ of a real variable, finds a companion function $H(u(t))$ in the same domain by convolution with the function $\frac{1}{\pi t}$ :

$$H(u(t)) = -\frac{1}{\pi}\vartheta.p.\int\limits_{-\infty}^{+\infty} \frac{u(\tau)}{t-\tau}d\tau.$$

The Kramers–Kronig relation applied to physics can be written as follows: for any complex function $f(\omega)$ of a complex variable $\omega$,

$$f(\omega) = u(\omega) + iv(\omega),$$

if the function $f(\omega)$ is analytic in the upper half-plane, $\omega \in \mathbb{R}_+$ and the following condition is fulfilled:

$$\lim_{|\omega|\to 0} f(\omega) \to 0,$$

then the Kramers–Kronig relation takes the form:

$$u(\omega) = 1 + \frac{1}{\pi}\vartheta.p. \int\limits_{-\infty}^{+\infty} \frac{\varepsilon''(\omega')}{\omega' - \omega} d\omega',$$

$$v(\omega) = -\frac{1}{\pi}\vartheta.p. \int\limits_{-\infty}^{+\infty} \frac{u(\omega')}{\omega' - \omega} d\omega'.$$

Here $\vartheta.p.$ denote taking the integral in the sense of the Cauchy principal value. It can be seen that the functions $u(\omega)$ and $v(\omega)$ are not independent, therefore, the complete function can be restored if only its real or imaginary part is given. The principal value of the Cauchy integral is a generalization of the concept of the Riemann integral, which allows to calculate some divergent improper integrals.

The main idea of the Cauchy principal value of the integral is that when the integration intervals approach the singular point from both sides at the same speed, the singularities level out each other (due to different signs on the left and right), and as a result, we can obtain a finite boundary. This boundary is called the Cauchy principal value of the integral.

The permittivity of a dispersed medium can be represented as a complex function of frequency:

$$\varepsilon(\omega) = \varepsilon'(\omega) + i\varepsilon''(\omega), \tag{1}$$

where $\varepsilon'(\omega)$ and $\varepsilon''(\omega)$ are real and imaginary parts of the permittivity, respectively. Then we can write the Kramers–Kronig relation for the permittivity as follows:

$$\varepsilon'(\omega) = 1 + \frac{1}{\pi}\vartheta.p. \int\limits_{-\infty}^{+\infty} \frac{\varepsilon''(x)}{x - \omega} dx,$$

$$\varepsilon''(\omega) = -\frac{1}{\pi}\vartheta.p. \int\limits_{-\infty}^{+\infty} \frac{\varepsilon'(x)}{x - \omega} dx.$$

However, known papers present no numerical verification of these relations. Since the calculation of the electrodynamic parameters is based on the results of measurements of the $S$-scattering matrix, therefore, we prove that for these measured values, the Kramers–Kronig relations are valid for such line operating conditions as short circuit, open circuit, and matched load.

## 2. Short Circuit

Let us prove that the Kramers–Kronig relation holds for the $S$-matrix measured under the short-circuit conditions. The short-circuit $S$-parameter can be represented as

$$S(\omega) = S'(\omega) + iS''(\omega), \tag{2}$$

where $S'(\omega)$ is a real part, $S''(\omega)$ is an imaginary part. Under the short-circuit conditions, the real part is $S'(\omega) = -1$, within the full frequency range, while the imaginary part is $S''(\omega) = 0$. Let us introduce the function $f(\omega) = S(\omega) + 1$.

**Lemma 1.** *Let $f = f(\omega)$, where $f(\omega) = S(\omega)+1$ and $S(\omega) = S'(\omega)+iS''(\omega)$ are measured values of the short-circuit $S$-parameters, and suppose that the following conditions hold:*
*(i) $f = f(\omega)$ is analytic in the upper half-plane $\mathbb{R}_+$;*
*(ii) $\lim\limits_{|\omega|\to 0} f(\omega) \to 0$.*
*Then $S(\omega)$ can be represented as*

$$S'(\omega) = \frac{1}{\pi}\vartheta.p.\int\limits_{-\infty}^{+\infty} \frac{S''(\omega')}{\omega' - \omega} d\omega' - 1,\ S''(\omega) = -\frac{1}{\pi}\vartheta.p.\int\limits_{-\infty}^{+\infty} \frac{S'(\omega')}{\omega' - \omega} d\omega'.$$

*Proof.* Step 1. Let us prove that the function $f(\omega)$ must be analytic. Since the function $S(\omega)$ is analytic, the Cauchy–Riemann relation holds:

$$\frac{\partial S'}{\partial x} = \frac{\partial S''}{\partial y},\ \frac{\partial S'}{\partial y} = -\frac{\partial S''}{\partial x}.$$

For the function $f(\omega)$ the real part is $\mathrm{Re}\ (f(\omega)) = S'(\omega) + 1$, and the imaginary part is $\mathrm{Im}\,(f(\omega)) = S''(\omega)$.

Let us calculate the derivative of the real part:

$$\frac{\partial}{\partial y}\mathrm{Re}\ (f(\omega)) = \frac{\partial S'}{\partial y} + \frac{\partial S''}{\partial y} = \frac{\partial S'}{\partial y} = -\frac{\partial S''}{\partial x}.$$

Since the Cauchy-Riemann relation holds, the function is also analytic.

Step 2. Let us prove that the condition $f(\omega) \to 0$ is satisfied at $\omega \to 0$.

Let us calculate $\lim\limits_{\omega\to 0} f(\omega)$. By the theorem on the limit of a complex function, the value $\psi_0 = u_0 + iv_0$ is the limit of the function $f(z) = u(x, y) + iv(x, y)$ at $z \to z_0 \Leftrightarrow \lim\limits_{\substack{x\to x_0\\ y\to y_0}} u(x, y) = u_0$ and $\lim\limits_{\substack{x\to x_0\\ y\to y_0}} v(x, y) = v_0$. For the real part $\mathrm{Re}\ (f(\omega))$, the limit is:

$$\lim_{\omega\to 0} f(\omega) = \lim_{\omega\to 0}[S'(\omega) + 1] = 0.$$

For the imaginary part $\mathrm{Im}\,(f(\omega))$

$$\lim_{\omega\to 0} \mathrm{Im}\,(f(\omega)) = \lim_{\omega\to 0} S''(\omega) = 0.$$

Accordingly, for the function $f(\omega)$, the relation $\lim\limits_{\omega\to 0} f(\omega) = 0$ is fulfilled.

$$\lim_{\omega\to 0} f(\omega) = \lim_{\omega\to 0} \mathrm{Re}\ (f(\omega)) + \lim_{\omega\to 0} \mathrm{Im}\,(f(\omega)) = 0.$$

Since the function $f(\omega)$ is analytic, and its limit is equal to 0, the conditions for the Kramers–Kronig relation are satisfied:

$$\mathrm{Re}\ (f(\omega)) = \frac{1}{\pi}\vartheta.p.\int\limits_{-\infty}^{+\infty} \frac{\mathrm{Im}\,(f(\omega')}{\omega' - \omega} d\omega',\ \mathrm{Im}\,(f(\omega)) = -\frac{1}{\pi}\vartheta.p.\int\limits_{-\infty}^{+\infty} \frac{\mathrm{Re}\ (f(\omega')}{\omega' - \omega} d\omega'.$$

$$S'(\omega)+1=\frac{1}{\pi}\vartheta.p.\int\limits_{-\infty}^{+\infty}\frac{S''(\omega'(\omega')}{\omega'-\omega}d\omega',\ S''(\omega)=-\frac{1}{\pi}\vartheta.p.\int\limits_{-\infty}^{+\infty}\frac{(S'(\omega')+1)(\omega')}{\omega'-\omega}d\omega'.$$

$$S'(\omega)=\frac{1}{\pi}\vartheta.p.\int\limits_{-\infty}^{+\infty}\frac{S''(\omega'(\omega')}{\omega'-\omega}d\omega'-1,\ S''(\omega)=-\frac{1}{\pi}\vartheta.p.\int\limits_{-\infty}^{+\infty}\frac{(S'(\omega')+1)(\omega')}{\omega'-\omega}d\omega'.$$

## 3. Open Circuit

Let us prove that the Kramers–Kronig relation holds for the open-circuit $S$-matrix measurements. The $S$-parameter for open circuit can be represented as (2) where $S'(\omega)$ is a real part, $S''(\omega)$ is an imaginary part. For open circuit, the real part is $S'(\omega)=1$, within the full frequency range, while the imaginary part is $S''(\omega)=0$.

**Lemma 2.** *Let $f=f(\omega)$, where $f(\omega)=S(\omega)-1$ and $S(\omega)=S'(\omega)+iS''(\omega)$ are measured values of the open-circuit S-parameters, and the following conditions hold:*

*(i) $f=f(\omega)$ is analytic in the upper half-plane $\mathbb{R}_+$;*

*(ii) $\lim\limits_{\omega\to 0}f(\omega)\to 0$.*

*Then $S(\omega)$ can be represented as:*

$$S'(\omega)=\frac{1}{\pi}\vartheta.p.\int\limits_{-\infty}^{+\infty}\frac{S''(\omega'))}{\omega'-\omega}d\omega'+1,S''(\omega)=-\frac{1}{\pi}\vartheta.p.\int\limits_{-\infty}^{+\infty}\frac{S'(\omega'))}{\omega'-\omega}d\omega'.$$

*Proof.* Step 1. Let us prove that the function $f(\omega)=S(\omega)-1$ is analytic. Since the function $S(\omega)$ is analytic, the Cauchy–Riemann relation holds:

$$\frac{\partial S'}{\partial x}=\frac{\partial S''}{\partial y},\ \frac{\partial S'}{\partial y}=-\frac{\partial S''}{\partial x}.$$

For the function $f(\omega)$, the real part is $\mathrm{Re}\,(f(\omega))=S'(\omega)-1$, and the imaginary part is $\mathrm{Im}\,(f(\omega))=S''(\omega)$. The derivative of the real part is

$$\frac{\partial}{\partial y}\,\mathrm{Re}\ (f(\omega))=\frac{\partial S'}{\partial y}-\frac{\partial S''}{\partial y}=\frac{\partial S'}{\partial y}=-\frac{\partial S''}{\partial x}.$$

Since the Cauchy–Riemann relation holds, the function $f(\omega)$ is also analytic.

Step 2. Let us prove that at $\omega\to 0$ , the condition $f(\omega)\to 0$ is satisfied.

Let us calculate $\lim\limits_{\omega\to 0}f(\omega)$. By the theorem on the limit of a complex function, the value $\psi_0=u_0+iv_0$ is the limit of the function $f(z)=u(x,y)+iv(x,y)$ at $z\to z_0\Leftrightarrow \lim\limits_{\substack{x\to x_0\\ y\to y_0}}u(x,y)=u_0$ and $\lim\limits_{\substack{x\to x_0\\ y\to y_0}}v(x,y)=v_0$. For the real part $\mathrm{Re}\ (f(\omega))$, the limit is

$$\lim_{\omega\to 0}f(\omega)=\lim_{\omega\to 0}[S'(\omega)-1]=0.$$

For the imaginary part $\mathrm{Im}\,(f(\omega))$,

$$\lim_{\omega\to 0}\mathrm{Im}\,(f(\omega))=\lim_{\omega\to 0}S''(\omega)=0.$$

Accordingly, for the function $f(\omega)$, the condition $\lim\limits_{\omega\to 0} f(\omega)=0$ is fulfilled.

$$\lim_{\omega\to 0} f(\omega)=\lim_{\omega\to 0}\mathrm{Re}\,(f(\omega))+\lim_{\omega\to 0}\mathrm{Im}\,(f(\omega))=0.$$

Since the function $f(\omega)$ is analytic, and its limit is equal to 0, the conditions for the Kramers–Kronig relation are satisfied:

$$\mathrm{Re}\,(f(\omega))=\frac{1}{\pi}\vartheta.p.\int\limits_{-\infty}^{+\infty}\frac{\mathrm{Im}\,(f(\omega'))}{\omega'-\omega}d\omega',\ \mathrm{Im}\,(f(\omega))=-\frac{1}{\pi}\vartheta.p.\int\limits_{-\infty}^{+\infty}\frac{\mathrm{Re}\,(f(\omega'))}{\omega'-\omega}d\omega'.$$

$$S'(\omega)-1=\frac{1}{\pi}\vartheta.p.\int\limits_{-\infty}^{+\infty}\frac{(S''(\omega'))(\omega')}{\omega'-\omega}d\omega',\ S''(\omega)=-\frac{1}{\pi}\vartheta.p.\int\limits_{-\infty}^{+\infty}\frac{(S'(\omega')+1)(\omega')}{\omega'-\omega}d\omega'.$$

$$S'(\omega)=\frac{1}{\pi}\vartheta.p.\int\limits_{-\infty}^{+\infty}\frac{S''(\omega')}{\omega'-\omega}d\omega'+1,\ S''(\omega)=-\frac{1}{\pi}\vartheta.p.\int\limits_{-\infty}^{+\infty}\frac{S'(\omega')}{\omega'-\omega}d\omega'.$$

□

## 4. Matched Load

Let us prove that the Kramers–Kronig relation holds for the S-matrix measured under the matched load conditions, for which the real part is $S'(\omega)=0$, within the full frequency range, while the imaginary part $S''(\omega)=0$.

**Lemma 3.** *Consider the measured values of the S-parameters, where $S(\omega)=S'(\omega)+iS''(\omega)$, under the matched load, and assume that the following conditions hold:*
*(i) $S=S(\omega)$ is analytic in the upper half-plane $\mathbb{R}_+$;*
*(ii) for the function $S=S(\omega)$, the condition $\lim\limits_{|\omega|\to 0} f(\omega)\to 0$ is satisfied.*

*Then, $S(\omega)$ can be represented as:*

$$S'(\omega)=\frac{1}{\pi}\vartheta.p.\int\limits_{-\infty}^{+\infty}\frac{S''(\omega')}{\omega'-\omega}d\omega',\ S''(\omega)=-\frac{1}{\pi}\vartheta.p.\int\limits_{-\infty}^{+\infty}\frac{S'(\omega')}{\omega'-\omega}d\omega'.$$

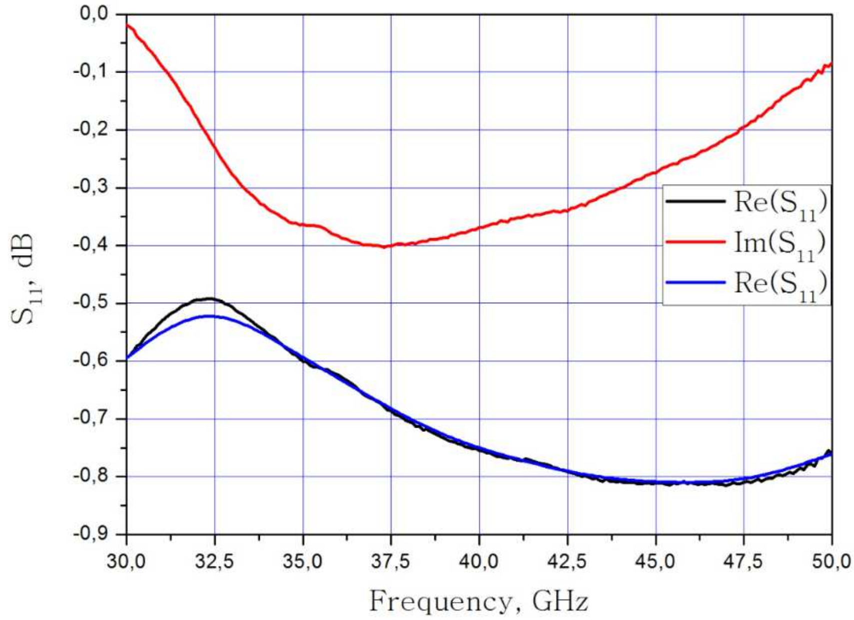
*Proof.* Based on Lemmas 1 and 2 and the initial data $S'(\omega)=0, S''(\omega)=0$, we obtain:

$$\mathrm{Re}\,(f(\omega))=\frac{1}{\pi}\vartheta.p.\int\limits_{-\infty}^{+\infty}\frac{\mathrm{Im}\,(f(\omega'))}{\omega'-\omega}d\omega',\ \mathrm{Im}\,(f(\omega))=-\frac{1}{\pi}\vartheta.p.\int\limits_{-\infty}^{+\infty}\frac{\mathrm{Re}\,(f(\omega'))}{\omega'-\omega}d\omega'.$$

Since $\mathrm{Re}\,(f(\omega))=S'(\omega')$ and $\mathrm{Im}\,(f(\omega))=S''(\omega)$, we get:

$$S'(\omega)=\frac{1}{\pi}\vartheta.p.\int\limits_{-\infty}^{+\infty}\frac{S''(\omega')}{\omega'-\omega}d\omega',\ S''(\omega)=-\frac{1}{\pi}\vartheta.p.\int\limits_{-\infty}^{+\infty}\frac{S'(\omega')}{\omega'-\omega}d\omega'.$$

□




Frequency dependence of the real and imaginary parts of $S_{11}$

## 5. Experimental Data

Based on the measured values of S-parameters for carbonyl iron, the real part of $S_{11}$ was calculated from the measured values of the imaginary part of $S_{11}$. The results of measurements and calculations are shown in Figure. The black line is the measured values, the blue line is calculated value.

Based on the obtained measurement results, we can propose the following algorithm for verifying the measured values.

1. Measure $S$-parameters and input resistance for line operation under short-circuit, open-circuit, and matched load conditions.

2. Check the fulfillment of the Kramers–Kronig relation for the measured values of S-parameters (formulas (3) and (4)) under the short circuit conditions. If the relations are not satisfied, check the meter calibration under the short circuit conditions.

3. Check the fulfillment of the Kramers–Kronig relation for the measured values of S-parameters (formulas (5) and (6)) under the open-circuit conditions.

4. Check the fulfillment of the Kramers–Kronig relation for the measured values of S-parameters (formulas (7) and (8)) under the matched load conditions.

5. After completing Steps 1 – 4 and checking the fulfillment of the relations for S-parameters in all the measurement conditions, calculate the electrodynamic parameters: permittivity and permeability.

6. Check the fulfillment of the Kramers–Kronig relation for the permittivity and permeability (formulas (13) – (16)).

7. From the calculated values of the permittivity and permeability, it is possible to calculate the input resistance of the loaded line. Check the obtained calculated value of the input resistance with the measured input resistance in the matched load mode in Step 1.

## Conclusions

This paper showed that the experimental values of the $S$-parameters for different operating modes of the line satisfy the Kramers–Kronig relation. The Kramers–Kronig

relations express the relationship between the real and imaginary parts of the permittivity in classical electrodynamics. We proved lemmas for various line operating conditions and developed the algorithm for verifying the measurements based on the proved lemmas. Comparison of theoretical and experimental values showed their good quantitative and qualitative agreement.

**Acknowledgements.** *This work was carried out in South Ural State University (National Research University) with a financial support of the Russian Science Foundation (project no. 21-79-10115, https://rscf.ru/project/21-79-10115/).*

## References

1. Kronig R. de L. On the Theory of the Dispersion of X-Rays. *Journal of the Optical Society of America*, 1926, no. 12, pp. 547–557.
2. Kramers H.A. La diffusion de la lumiere par les atomes. *Transactions of Volta Centenary Congress*, 1927, no. 2, pp. 545–557. (in French)
3. Landau L.D., Lifshits E.M. *Elektrodinamika sploshnyh sred* [Electrodynamics of Continuous Media]. Moscow, Nauka, 1992. (in Russian)
4. Ponomarenko V.I., Lagunov I.M. *Poglotiteli elektromagnitnyh voln. Radiofizicheskaya teoriya* [Absorbers of Electromagnetic Waves. Radiophysical Theory. Methods of Calculation]. Simferopol, Polyprint, 2021. (in Russian)
5. Toll J.S. Causality and the Dispersion Relation: Logical Foundations. *Physical Review*, 1956, no. 104, pp. 1760–1770.
6. King F.W. *Hilbert Transforms.* Cambridge, Cambridge University Press, 2009.
7. Bracewell R. *The Hilbert Transform. The Fourier Transform and its Applications.* New York, McGraw-Hill, 1999, pp. 267–272.
8. Papoulis A. *Hilbert Transforms. The Fourier Integral and its Applications.* New York, McGraw-Hill, 1962, pp. 198–201.
9. Balagurov B.Ya. *Elektrofizicheskie svoystva kompozitov. Makroskopicheskaya teoriya* [Electrophysical Properties of Composites. Macroscopic Theory]. Moscow, Lenand, 2018. (in Russian)

*Received March 5, 2022*

---

**УДК 537.874.4** **DOI: 10.14529/mmp220310**

## АЛГОРИТМ ПРОВЕРКИ АДЕКВАТНОСТИ ИЗМЕРЕНИЙ

***А.Л. Шестаков*[1], *Д.С. Клыгач*[1], *М.Г. Вахитов*[1]**
[1]Южно-Уральский государственный университет, г. Челябинск,
Российская Федерация

В статье приведено описание соотношения Крамерса – Кронига для проверки полученных значений $S$-параметров при различных режимах работы линии передачи. Получены и доказаны леммы для $S$-параметров режимов работы линии при коротком замыкании, холостом ходе и согласованной нагрузке. Приведено сравнение теоретических и экспериментальных значений, которые подтверждают правильность полученных соотношений и выводов.

*Ключевые слова: соотношение Крамерса – Кронига; измерение; электродинамические параметры; проверка измерений.*

## Литература


1. de L. Kronig, R. On the Theory of the Dispersion of X-Rays / R. de L. Kronig // Journal of the Optical Society of America. – 1926. – №12. – P. 547–557.
2. Kramers, H.A. La diffusion de la lumiere par les atomes / H.A. Kramers // Transactions of Volta Centenary Congress. – 1927. – № 2. – P. 545–557.
3. Ландау, Л.Д. Электродинамика сплошных сред / Л.Д. Ландау, Е.М. Лифшиц. – М.: Наука, 1992.
4. Пономаренко, В.И. Поглотители электромагнитных волн. Радиофизическая теория. Методы расчета / В.И. Пономаренко, И.М. Лагунов. – Симферополь: ПОЛИПРИНТ, 2021.
5. Toll, J.S. Causality and the Dispersion Relation: Logical Foundations / J.S. Toll // Physical Review. – 1956. – № 104. – P. 1760–1770.
6. King, F.W. Hilbert Transforms / F.W. King. – Cambridge: Cambridge University Press, 2009.
7. Bracewell, R. The Hilbert Transform. The Fourier Transform and Its Applications / R. Bracewell. – New York: McGraw-Hill, 1999. – P. 267–272.
8. Papoulis, A. Hilbert Transforms. The Fourier Integral and Its Applications / A. Papoulis. – New York: McGraw-Hill, 1962. – P. 198–201.
9. Балагуров, Б.Я. Электрофизические свойства композитов. Макроскопическая теория / Б.Я. Балагуров. – М: ЛЕНАНД, 2018.



Александр Леонидович Шестаков, доктор технических наук, профессор, кафедра информационно-измерительной техники, Южно-Уральский государственный университет (г. Челябинск, Российская Федерация), a.l.shestakov@susu.ru.

Денис Сергеевич Клыгач, кандидат технических наук, доцент, кафедра математического и компьютерного моделирования, Южно-Уральский государственный университет (г. Челябинск, Российская Федерация), klygachds@susu.ru.

Максим Григорьевич Вахитов, кандидат технических наук, доцент, кафедра конструирования и производства радиоаппаратуры, Южно-Уральский государственный университет (г. Челябинск, Российская Федерация), vmg@kipr.susu.ac.ru.


*Поступила в редакцию 5 марта 2022 г.*