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INVESTIGATION OF BOUNDARY CONTROL AND FINAL OBSERVATION IN MATHEMATICAL MODEL OF MOTION SPEED POTENTIALS DISTRIBUTION OF FILTERED LIQUID FREE SURFACE

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In this paper, we study the problem of boundary control and final observation for one degenerate mathematical model of motion speed potentials distribution of filtered liquid free surface with the Showalter–Sidorov initial condition. The mathematical model is based on the degenerate Boussinesq equation with an inhomogeneous Dirichlet condition. This model belongs to the class of semilinear Sobolev-type models in which the nonlinear operator is p-coercive and s-monotone. In the paper, the problem of boundary control and final observation for a semilinear Sobolev-type model is considered and conditions for the existence of a control-state pair of the problem are found. In applied studies of a research problem, it is allowed to find such a potentials distributionof filtered liquid free surface, at which the system transitions from the initial condition to a given final state within a certain period of time T.

Keywords: mathematical model of motion speed potentials distribution of filtered liquid free surface; problem of boundary control and final observation; the Sobolev type equations.

Introduction

The control of various processes is a key problem on a global scale, where scientific and technological advances are becoming an increasingly important role in people's daily lives. It is associated with areas including industry, transportation, energy, medicine, and many others. The implementation of effective process control requires a wide range of knowledge in the field of science, technology and control. In this regard, the constant development and improvement of methods and technologies for managing access to processes is a strategic problem for the development of science and technology in general.

The fundamental problem of regulation is to identify the influences on a system that results in a controlled change in the system's state. Additionally, the type of control implemented is a significant factor that influences the quality of the transient process. Keep in mind that the phrase "quality of the transient process " encompasses various parameters, like performance. For instance, the transition process duration (speed optimization) could serve as an assessment criterion for the control system. The examination of such problems acted as a prototype for the optimal control theory issue, where the control parameter refers to an external force acting on the system.

We highlight J.-L. Lions and A.V. Fursikov as significant contributors to the field of optimal control theory. For instance, the paper [1] systematically examines the optimal control problems associated with partial differential equations. In another notable study, A.V. Fursikov's work [2] demonstrates the existence of a solution to the problem of model control based on the Navier–Stokes and Euler equations. During the initial investigations, G.A. Sviridyuk and A.A. Efremov studied the optimal control problem for linear Sobolev type equations [3]. Subsequently, this problem was explored in various contexts employing nonlinear Sobolev type equations [4].

Another important type of control, i.e. start control, arises in a situation where the control parameters enter the initial conditions. Usually, in start-up control problems, the initial state of the system serves as the control, and the objective functional is the final one, i.e. only the final state of the system (final observation) is observed. In certain situations, it is crucial to observe the status of the managed system during intermediate intervals, while approximating the difference of current properties, such as velocity, temperature, pressure, etc., from their desired values. This problem has been the subject of many works: in [5] the existence of a solution to the problem under study for the Barenblat–Gilman model is proved; in [6] this type of control is considered for fluid filtration model.

Note also that the control can enter not only into the right-hand sides of the equations of state or initial conditions, but also into the boundary conditions. The term boundary control pertains to the identification of control functions, which facilitate the transition of the system from an initial state to a predetermined final state over a defined time period of T. Typically, boundary control is utilized in scenarios involving rod oscillation or mass and heat transfer, such as regulating the heat exchange process where controlling the change in heat flux entering the designated area leads to substantial changes. The existence of a boundary control for both parabolic and hyperbolic systems is presented in [7]. The approach developed by J.-L. Lions is applied to understand different physical processes as described in works such as [8, 9].

The purpose of this work is to study the problem of boundary control and final observation (search for a pair of states $(\hat{x}(T), \hat{u})$):

$$J(x(T),u) = \vartheta\|x(T,\cdot)-x_d(\cdot)\|^p_{L_p(D)} + (1-\vartheta)\|u-x_\Gamma\|^p_{L_p(0,T;W_p^{-\frac{1}{p}}(\Gamma))} \to \inf, \quad \vartheta \in (0,1), \tag{1}$$

for the mathematical model of motion speed potentials distribution of filtered liquid free surface, which is based on the Boussinesq equation

$$(\lambda - \Delta)x_t - \Delta(|x|^{p-2}x) = f, \ p \geq 2, \tag{2}$$

with the Showalter–Sidorov initial condition

$$(\lambda - \Delta)(x(0) - x_0) = 0 \tag{3}$$

and the inhomogeneous Dirichlet condition

$$x(s,t) = u(s,t), \ (s,t) \in \Gamma \times [0,T]. \tag{4}$$

For $p = 3$, equation (2) simulates the motion of the free surface of a liquid filtering in a porous medium [10]. For $n = 2, 3$, the desired function $x = x(s,t)$ ($s \in D$, D is a bounded domain in $\mathbb{R}^n$ with the boundary Γ of the class C^∞, $t \in [0,T]$) describes the change in the free surface motion speed potential, and the given function $f = f(s,t)$ describes fluid sources, the parameter $\lambda \in \mathbb{R}$ characterizes the rock [11]. Numerous research papers are dedicated to examining the possibility of solving initial-boundary value problems for equation (2). One such example is the work of [12], where they establish the existence and uniqueness of a classical solution to the first boundary value problem for equation (2). The article [13] explores equation (2) featuring a non-linear, non-constant, non-monotone source and demonstrates the possibility of solving the first initial-boundary value problem. Meanwhile, [14] provides a numerical solution algorithm for the generalized Boussinesq equation, which characterizes the movement of the fluid surface filter within a range of finite depth. In functional (1), the given functions $x_d(s)$ and $x_\Gamma(s,t)$ characterize the required state of the system at the final moment of time and the fixed state of the system at the boundary, respectively.

1. Mathematical Model

Next, we define function spaces as follows: $\mathcal{H} = (W_2^{-1}(D), < \cdot, \cdot >)$, $\mathfrak{H} = L_2(D)$, $\mathfrak{B} = L_p(D)$, moreover, $\mathfrak{B}^* = (L_p(D))^*$ and $\mathfrak{H}^* = (L_2(D))^*$ are dual spaces with $< \cdot, \cdot >$.

With this definition of $\mathfrak{H}^*$ and $\mathfrak{B}^*$, there are dense and continuous embeddings [11]:

$$\mathfrak{B} \hookrightarrow \mathfrak{H} \hookrightarrow \mathcal{H} \hookrightarrow \mathfrak{H}^* \hookrightarrow \mathfrak{B}^*.$$

Consider the homogeneous Dirichlet problem for an operator $(-\Delta)$: $-\Delta\varphi_k = \lambda_k\varphi_k$, $\varphi_k \in \overset{\circ}{W}{}_2^1(D)$. Note also that $\tilde{\varphi}_k{=}(-\Delta)^{-1}\varphi_k = \dfrac{\varphi_k}{\lambda_k}$. The investigation is based on the theory of solvability in a weakly generalized case of the Showalter–Sidorov problem

$$A(x(0) - x_0) = 0$$

for the semilinear equation

$$Ax_t + B(x) = g$$

in the case the operator A is linear, symmetric, continuous and non-negative definite, the operator B is p-coercive and s-monotone. Let's put

$$< g, v >= \int\limits_D f\tilde{v}ds - \int\limits_\Gamma |u|^{p-2}u\frac{\partial\tilde{v}}{\partial n}dS, \quad v \in \mathfrak{B}.$$

Define the operators A and B as follows:

$$\langle Ax, y\rangle = \int\limits_D (\lambda x\tilde{y} + xy)ds, \quad x,\, y \in \mathfrak{H};$$

$$\langle B(x), y\rangle = \int\limits_D |x|^{p-2}xyds, \quad x,\, y \in \mathfrak{B},$$

where $\tilde{y}$ is solution of the homogeneous Dirichlet problem in the domain D for equation $-\Delta y = \tilde{y}$ [15]. The paper [11] shows the properties of the operators A and B: for any $\lambda > -\lambda_1$, the operator $A \in \mathcal{L}\,(\mathfrak{N}, \mathfrak{N}^*)$ is self-adjoint, Fredholm, and non-negative definite; the operator $B \in C^\infty(\mathfrak{B}, \mathfrak{B}^*)$ is s-monotone and p-coercive. Next, let's consider the case of $\lambda \geq -\lambda_1$ and consider the set

$$\text{coim}\, A = \begin{cases} \mathfrak{H}, \text{ if } \lambda > -\lambda_1; \\ \{x \in \mathfrak{H} : \langle x, \varphi_1\rangle = 0\}, \text{ if } \lambda = -\lambda_1 \end{cases}$$

and

$$\mathfrak{M} = \begin{cases} \mathfrak{B}, \text{ if } \lambda > -\lambda_1; \\ \{x \in \mathfrak{B} : \int\limits_D |x|^{p-2}x\varphi_1\; ds + \int\limits_\Gamma |u|^{p-2}u\dfrac{\partial\tilde{\varphi}_1}{\partial n}dS = \int\limits_D f\tilde{\varphi}_1 ds, \} \text{ if } \lambda = -\lambda_1. \end{cases}$$

This set is a Banach C^1-manifold [15], which is diffeomorphic, except perhaps for the zero point, to the subspace $\{x \in L_p(D) : \int\limits_D x\varphi_1 ds = 0\}$ under the condition that $\left(\int\limits_\Gamma |u|^{p-2}u\dfrac{\partial\tilde{\varphi}_1}{\partial n}dS + \int\limits_D f\tilde{\varphi}_1 ds\right)$ does not depend on t in the case $\lambda = -\lambda_1$. The form in which we present the estimated solutions to the problem (2) – (4) is

$$x_k(s,t) = \sum_{i=1}^{k} c_i(t)\varphi_i(s), \quad k > \dim\; \ker A, \tag{5}$$

where the coefficients $c_i = c_i(t),\ i = 1, ..., k$, are determined by the system of equations

$$< Ax_t + B(x), \varphi_i > = < f, \varphi_i > \qquad i = 1, \ldots, k, \tag{6}$$

and the conditions

$$< A(x(0) - x_0), \varphi_i > = 0, \quad i = 1, .., k. \tag{7}$$

Consider the space

$$\mathfrak{X} = \{x \mid x \in L_\infty(0, T; \text{coim } A) \cap L_p(0, T; \mathfrak{B})\}.$$

Before formulating the theorem on the existence of a solution (2) – (4), it should be noted that we are considering a weak generalized solution the Showalter–Sidorov problem for the mathematical model of motion speed potentials distribution of filtered liquid free surface, described in [11].

Theorem 1. *Let $u \in L_p(0, T; W_p^{-\frac{1}{p}}(\Gamma)),\ f \in L_q(0, T; W_q^{-1}(D)),\ p \geq 2,\ \lambda \geq -\lambda_1$. Then $\forall x_0 \in \mathfrak{B}$ and $T > 0$ there exists the weak generalized solution $x \in \mathfrak{X}$ to problem* (2) – (4).

Proof. The proof of Theorem 1 is based on the monotonicity method and the phase space technique, which necessitates creating a priori estimates. In coim A we introduce the norm $\mid x \mid^2 = < Ax, x >$. It involves utilizing the Banach–Alaoglu theorem and transitioning to the weak limit to demonstrate that the required solution is found. The proof is analogous to that of the case of the homogeneous Dirichlet condition [4]: the equation (6) multiplied by $c_i,\ i = \overline{1, k}$, and integrate by $(0, t)$

$$\int_0^t \left(\int_D ((\lambda - \Delta) x_{k\tau} \tilde{x}_k + |x_k|^{p-2} x_k^2) ds \right) d\tau = \int_0^t \left(\int_D f \tilde{x}_k ds \right) d\tau - \int_\Gamma |u_k|^{p-2} u_k \frac{\partial \tilde{x}_k}{\partial n} dS \leq$$

$$\leq \int_0^t \| \tilde{x}_k \|_{\overset{\circ}{W}{}^1_p(D)} \| f \|_{W_q^{-1}(D)} d\tau - \int_\Gamma |u_k|^{p-2} u_k \frac{\partial \tilde{x}_k}{\partial n} dS.$$

The difference is only in the construction of the a priori estimate

$$|x|^2 + \| x \|^p_{L_p(0,T;\mathfrak{B})} \leq C(\| f \|^q_{L_q(0,T;W_q^{-1}(D))} + |x_0|^2 + \| u \|^p_{L_p(0,T;W_p^{-\frac{1}{p}}(\Gamma))}.$$

It is worth noting that Theorem 1 indicates how Galerkin approximations (5) approach a general weak solution to problem (2) – (4).

□

2. Problem of Boundary Control and Final Observation

We now shift our focus towards investigating the problem of boundary control and final observation concerning the motion speed potentials distribution in a filtering liquid. It is necessary to construct the control space

$$\mathfrak{U} = \{L_p(0, T; W_p^{-\frac{1}{p}}(\Gamma)) : \int_\Gamma |u|^{p-2} u \frac{\partial \varphi_1}{\partial n} dS + \int_D f \tilde{\varphi}_1 ds \text{ does not depend on } t \text{ at } \lambda = -\lambda_1\},$$

and choose a non-empty, closed, convex set $\mathfrak{U}_{ad} \subset \mathfrak{U}$. The solution of problem (1) – (4) is to find a pair of functions $(\hat{x}(T), \hat{u})$ that satisfies the following condition:

$$J(\hat{x}(T), \hat{u}) = \inf_{(x(T),u)} J(x(T), u),$$

where the pair $(\hat{x}, \hat{u}) \in \mathfrak{X} \times \mathfrak{U}_{ad}$ satisfies (2) – (4). By the set of admissible pairs $\mathfrak{A}$ of problem (1) – (4) we mean the set of such pairs $(x; u)$ satisfying problem (2) – (4) and $J(x(T), u) < +\infty$. If $\mathfrak{U}_{ad} = \varnothing$, then for all $u \in \mathfrak{U}_{ad} \subset \mathfrak{U}$ the set of admissible pairs $(x(T), u)$ is not empty.

Theorem 2. [16] Let the conditions of Theorem 1 be satisfied. Then $\forall x_0 \in \mathfrak{B}$ and $T > 0$ there exists the solution $(\hat{x}(T), \hat{u})$ to problem (1) – (4).

The proof of the Theorem 2 is based on the monotonicity method, compactness method, Mazur's theorem, passage to the weak limit and is carried out similarly to the proof presented in [Theorem 2.1, 16].

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ИССЛЕДОВАНИЕ ГРАНИЧНОГО УПРАВЛЕНИЯ И ФИНАЛЬНОГО НАБЛЮДЕНИЯ В МАТЕМАТИЧЕСКОЙ МОДЕЛИ РАСПРЕДЕЛЕНИЯ ПОТЕНЦИАЛОВ СКОРОСТИ ДВИЖЕНИЯ СВОБОДНОЙ ПОВЕРХНОСТИ ФИЛЬТРУЮЩЕЙСЯ ЖИДКОСТИ

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Статья посвящена исследованию задачи граничного управления и финального наблюдения для одной вырожденной математической модели распределения потенциалов скорости движения свободной поверхности фильтрующейся жидкости с начальным условием Шоуолтера – Сидорова. Математическая модель базируется на вырожденном уравнении Буссинеска с неоднородным условием Дирихле. Исследуемая модель относится к классу полулинейных моделей соболевского типа, в которых нелинейный оператор является p-коэрцитивным и s-монотонным. Найдены условия существования пары управление-состояние изучаемой задачи. В прикладных исследованиях решение данной задачи позволяет находить такое распределение потенциалов скорости фильтрующейся жидкости, при котором происходит переход системы из начального состояния в заданное конечное состояние с течением определенного периода времени T.

Ключевые слова: задача граничного управления и финального наблюдения; математическая модель распределения потенциалов скорости движения свободной поверхности фильтрующейся жидкости; уравнения соболевского типа.

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