

TRAVELLING BREAKING WAVES

N.M. Koshkarbayev^{1,2}¹Institute of Mathematics and Mathematical Modeling, Almaty, Kazakhstan²Al-Farabi Kazakh National University, Almaty, Kazakhstan

E-mail: nurbol-koshkarbaev@mail.ru

We study a mathematical model of coastal waves in the shallow water approximation. The model contains two empirical parameters. The first one controls turbulent dissipation. The second one is responsible for the turbulent viscosity and is determined by the turbulent Reynolds number. We study travelling waves solutions to this model. The existence of an analytical and numerical solution to the problem in the form of a traveling wave is shown. The singular points of the system are described. It is shown that there exists a critical value of the Reynolds number corresponding to the transition from a monotonic profile to an oscillatory one. The paper is organized as follows. First, we present the governing system of ordinary differential equations (ODE) for travelling waves. Second, the Lyapunov function for the corresponding ODE system is derived. Finally, the behavior of the solution to the ODE system is discussed.

Keywords: shallow-water equation; Lyapunov function; Reynolds number; travelling wave solution.

Introduction

The theory of surface gravity waves is a classical branch of hydrodynamics [1–4]. However, the modelling of the wave breaking remains a difficult subject [5–7]. We study a recent model of breaking waves proposed in [8, 9]. The equations are derived by using the depth averaging of the equations for mass, momentum and kinetic energy. The model describes the wave breaking phenomenon by introduction of a new variable that is enstrophy (squared vorticity) following the works [10–12]. However, compared to the above mentioned papers, the enstrophy generation is not related to the shock formation, but rather to the turbulent dissipation. The model is mathematically simpler than two-fluid models describing a fine structure of the vorticity formation and propagation developed further in [13] and generalized to the case of stratified fluids in [14].

The model contains two empirical parameters. The first one controls turbulent dissipation. The second one is responsible for the eddy viscosity and is determined by the turbulent Reynolds number. We study travelling waves solutions to this model.

The paper is organized as follows. In Section 2, we present the governing system of ordinary differential equations (ODE) for travelling waves. In Section 3, the Lyapunov function for the corresponding ODE system is derived, and in Section 4 the behavior of the solution to the ODE system is discussed.

1. Problem Statement

The following system of equations is established in [8]:

$$\begin{cases} \frac{\partial h}{\partial t} + \frac{\partial hU}{\partial x} = 0, \\ \frac{\partial hU}{\partial t} + \frac{\partial}{\partial x} \left(hU^2 + \frac{gh^2}{2} + h^3\varphi + \frac{h^2\ddot{h}}{3} \right) = \frac{\partial}{\partial x} \left(\frac{4}{Re} h^3 \sqrt{\varphi} \frac{\partial U}{\partial x} \right), \\ \frac{\partial h\varphi}{\partial t} + \frac{\partial hU\varphi}{\partial x} = \frac{8h\sqrt{\varphi}}{Re} \left(\frac{\partial U}{\partial x} \right)^2 - C_r h\varphi^{\frac{3}{2}}. \end{cases} \quad (1)$$

Here h is the water depth, U is the depth averaged horizontal velocity, g is the gravity constant and φ is the enstrophy variable. The system depends on two parameters: Re that is the Reynolds number and C_r that is the turbulence dissipation parameter. In system (1), the first equation is the conservation of mass, the second one is the conservation of momentum, and the third one is the enstrophy equation. The “dot” means the material derivative:

$$\dot{h} = \frac{\partial h}{\partial t} + U \frac{\partial h}{\partial x}, \quad \ddot{h} = \left(\frac{\partial}{\partial t} + U \frac{\partial}{\partial x} \right) \dot{h}.$$

We are looking for travelling wave solutions to (1). Let us introduce the travelling wave coordinate $\xi = x - Dt$. Here $D = \text{const}$ is the wave propagation velocity. The time and space derivatives are transformed as follows:

$$\begin{cases} \frac{\partial}{\partial x} \rightarrow \frac{d}{d\xi}, \\ \frac{\partial}{\partial t} \rightarrow -D \frac{d}{d\xi}. \end{cases}$$

From the mass conservation equation we get: $h(U - D) = m$. Without loss of generality, we can suppose that $m > 0$, i. e. the flow propagates through the front from left to the right. Then

$$\dot{h} = (U - D)h',$$

and

$$\ddot{h} = \frac{m^2}{h} \left(\frac{h'}{h} \right)',$$

where $h' = \frac{dh}{d\xi}$. Using these expressions, we reduce the momentum equation to the following form:

$$m \frac{dU}{d\xi} + \frac{d}{d\xi} \left(\frac{gh^2}{2} + h^3\varphi + \frac{h^2 m^2}{3 h} \left(\frac{h'}{h} \right)' \right) - \frac{d}{d\xi} \left(\frac{4}{Re} h^3 \sqrt{\varphi} \frac{dU}{d\xi} \right) = 0.$$

It is important to note that system (1) is invariant (i.e. it retains its form) under the Galilean transformation:

$$x = \tilde{x} + Vt,$$

$$U = \tilde{U} + V,$$

$$\tilde{t} = t$$

with

$$V = \text{const}.$$

Therefore, without loss of generality, we can consider the case where the wave speed of travelling waves is $D = 0$, i.e. the solution is stationary. Integrating once with respect to ξ , we obtain

$$\frac{m^2}{h} + \frac{gh^2}{2} + h^3\varphi + \frac{h^2 m^2}{3 h} \left(\frac{h'}{h} \right)' + \frac{4}{Re} h^2 m \sqrt{\varphi} \left(\frac{h'}{h} \right) = I = \text{const}.$$

This equality is equivalent to the following equation:

$$\left(\frac{h'}{h}\right)' = \frac{3I}{hm^2} - \frac{3}{h^2} - \frac{3gh}{2m^2} - \frac{3h^2\varphi}{m^2} - \frac{12h\sqrt{\varphi}}{Rem} \left(\frac{h'}{h}\right).$$

For convenience, we introduce the variable Q by $h = e^Q$. The equation of momentum can be written as follows:

$$P' = \frac{3I}{m^2}e^{-Q} - 3e^{-2Q} - \frac{3g}{2m^2}e^Q - \frac{3\varphi}{m^2}e^{2Q} - \frac{12P\sqrt{\varphi}}{Rem}e^Q,$$

$$Q' = P.$$

The equations for enstrophy becomes:

$$m\varphi' = \frac{8h\sqrt{\varphi}}{Re} \left(-\frac{m}{h^2}h'\right)^2 - C_r h\varphi^{\frac{3}{2}},$$

or

$$\varphi' = \frac{8m\sqrt{\varphi}}{Reh} \left(\frac{h'}{h}\right)^2 - \frac{C_r h\varphi^{\frac{3}{2}}}{m}.$$

It is easy to show that the above equality is equivalent to

$$\varphi' = \frac{8mP^2\sqrt{\varphi}}{Re}e^{-Q} - \frac{C_r\varphi^{\frac{3}{2}}}{m}e^Q.$$

Therefore, system (1) can be written as

$$\begin{cases} Q' = P, \\ P' = \frac{3I}{m^2}e^{-Q} - 3e^{-2Q} - \frac{3g}{2m^2}e^Q - \frac{3\varphi}{m^2}e^{2Q} - \frac{12P\sqrt{\varphi}}{Rem}e^Q, \\ \varphi' = \frac{8mP^2\sqrt{\varphi}}{Re}e^{-Q} - \frac{C_r\varphi^{\frac{3}{2}}}{m}e^Q. \end{cases} \quad (2)$$

In the case when $\varphi \equiv 0$, system (2) is Hamiltonian, i.e.,

$$\begin{cases} Q' = P, \\ P' = \frac{3I}{m^2}e^{-Q} - 3e^{-2Q} - \frac{3g}{2m^2}e^Q. \end{cases} \quad (3)$$

The Hamiltonian of system (3) is defined by

$$H(Q, P) = \frac{P^2}{2} - \frac{3}{2}e^{-2Q} + \frac{3g}{2m^2}e^Q + \frac{3I}{m^2}e^{-Q}.$$

Indeed, we can check that

$$\begin{cases} \frac{dQ}{d\xi} = \frac{\partial H}{\partial P}, \\ \frac{dP}{d\xi} = -\frac{\partial H}{\partial Q}. \end{cases}$$

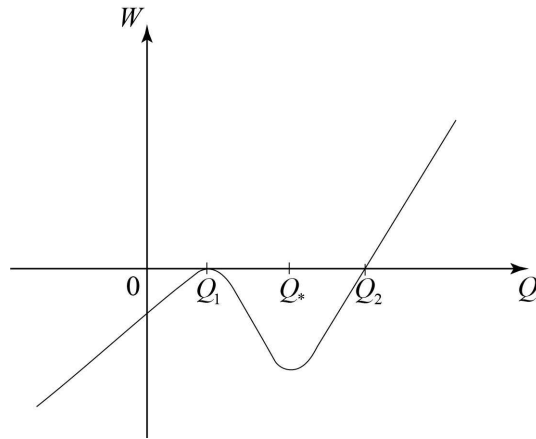


Fig. 1. The graph of the potential energy $W(Q)$

Let us present the Hamiltonian as the sum of kinetic and potential energy

$$H(Q, P) = \frac{P^2}{2} + W(Q),$$

where

$$W(Q) = -\frac{3}{2}e^{-2Q} + \frac{3g}{2m^2}e^Q + \frac{3I}{m^2}e^{-Q}.$$

The graph of the potential energy $W(Q)$ is shown in Fig. 1. We introduce the critical points Q_1 and Q_* of $W(Q)$:

$$\frac{\partial W}{\partial Q} \Big|_{Q=Q_1} = 0,$$

and

$$\frac{\partial W}{\partial Q} \Big|_{Q=Q_*} = 0.$$

Without loss of generality, we can assume that $W(Q_1) = 0$. Indeed, if $W(Q_1) = W_1 \neq 0$, we can replace $W(Q)$ by

$$W(Q) = -\frac{3}{2}e^{-2Q} + \frac{3g}{2m^2}e^Q + \frac{3I}{m^2}e^{-Q} - W_1.$$

To justify the behaviour of $W(Q)$ shown in Fig. 1, we formulate the properties of $W(Q)$ in the following proposition.

Proposition 1. *Let $Q_1 < Q_*$ (or, equivalently, $h_1 < h_*$) be the critical points of $W(Q)$. Then $|U_1| > \sqrt{gh_1}$ and $|U_*| < \sqrt{gh_*}$. It means that the flow ahead of the front (at $h = h_1$) is supercritical and behind the front (at $h = h_*$) is subcritical. Moreover, $\frac{\partial^2 W}{\partial Q^2} \Big|_{Q=Q_1} < 0$, and $\frac{\partial^2 W}{\partial Q^2} \Big|_{Q=Q_*} > 0$.*

Proof. Denote by $M_1 = (Q_1, 0)$ and $M_* = (Q_*, 0)$ the singular points of system (4). Then

$$\frac{\partial W}{\partial Q} = \frac{3}{m^2} \exp(-Q) \left(m^2 \exp(-Q) + \frac{g \exp(2Q)}{2} - I \right).$$

Replacing again $\exp(Q) = h$, we get that in the critical points

$$\frac{\partial W}{\partial Q} = \frac{3}{m^2 h} \left(\frac{gh^2}{2} + \frac{m^2}{h} - I \right) = 0.$$

In particular, this implies

$$\frac{gh_1^2}{2} + \frac{m^2}{h_1} = \frac{gh_*^2}{2} + \frac{m^2}{h_*}$$

or

$$(h_* - h_1) \left(\frac{m^2}{h_1 h_*} - \frac{gh_*}{2} - \frac{gh_1}{2} \right) = 0.$$

Then $h_* > h_1$ and $\frac{m^2}{h_1 h_*} = \frac{gh_*}{2} + \frac{gh_1}{2}$ imply that

$$gh_* > \frac{gh_*}{2} + \frac{gh_1}{2} = \frac{m^2}{h_1 h_*} > \frac{m^2}{h_*^2},$$

which implies $|U_*| < \sqrt{gh_*}$. Finally, the inequality

$$gh_1 < \frac{gh_*}{2} + \frac{gh_1}{2} = \frac{m^2}{h_1 h_*} < \frac{m^2}{h_1^2},$$

implies $|U_1| > \sqrt{gh_1}$. The inequalities for the second derivatives are direct consequences of the inequalities because $\text{sign} \frac{\partial^2 W}{\partial Q^2} \Big|_{Q=Q_i} = -\text{sign}(U_i^2 - gh_i)$, $i = 1, *$. The proposition is proved. □

This proposition establishes the behaviour of the function $W(Q)$ shown in Fig. 1.

2. Lyapunov Function

The Lyapunov function for system (3) is derived from the energy equation. Let us write down the energy equations [8]:

$$\frac{\partial h e}{\partial t} + \frac{\partial}{\partial x} \left(h U e + U \left(h^3 \varphi + \frac{gh^2}{2} + \frac{h^2 \ddot{h}}{3} - 4\nu_T h \frac{\partial U}{\partial x} \right) \right) = -h \ll \varepsilon \gg .$$

Here $e = \frac{U^2}{2} + \frac{h^2 \varphi}{2} + \frac{\dot{h}^2}{6} + \frac{gh}{2}$. For travelling wave solutions the energy equation takes the form:

$$\left(\frac{m^3}{6} \left(\frac{h'}{h} \right)^2 + \frac{mI}{h} + \frac{m\varphi h^2}{2} + \frac{ghm}{2} - \frac{m^3}{2h^2} \right)' = -h \ll \varepsilon \gg$$

where $\ll \varepsilon \gg = \frac{C_r}{2} h^2 \varphi^{3/2}$. Therefore, the Lyapunov function for system (3) is

$$L(Q, P, \varphi) = \frac{m^3}{6} P^2 + mI e^{-Q} + \frac{m\varphi}{2} e^{2Q} + \frac{gm}{2} e^Q - \frac{m^3}{2} e^{-2Q}.$$

The graph of the Lyapunov function is shown in Fig. 2.

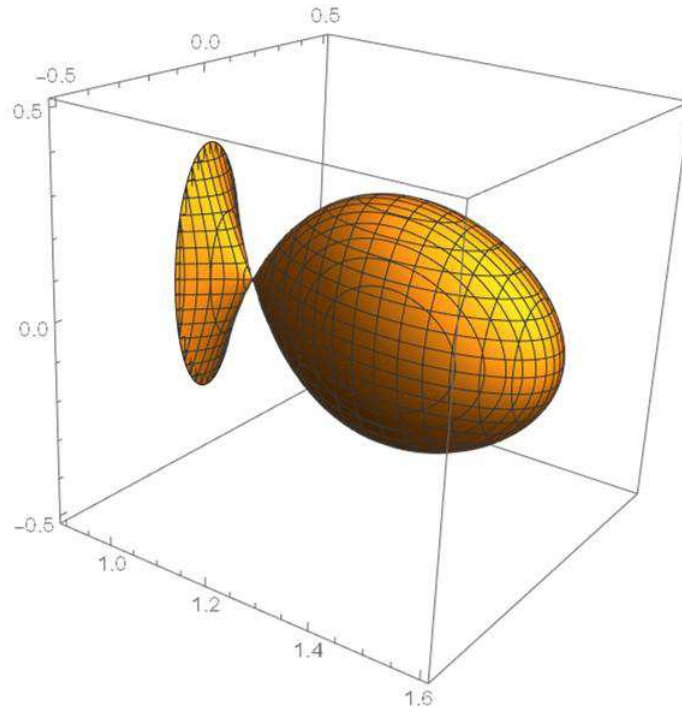


Fig. 2. The graph of the Lyapunov function

3. Existence of Travelling Wave Solutions

Denote $\Omega : \varphi = \Omega^2$. Then system (3) takes the form

$$\begin{cases} Q' = P, \\ P' = \frac{3I}{m^2}e^{-Q} - 3e^{-2Q} - \frac{3g}{2m^2}e^Q - \frac{3\Omega^2}{m^2}e^{2Q} - \frac{12P|\Omega|}{mRe}e^Q, \\ \Omega' = \frac{4mP^2 \operatorname{sgn}(\Omega)}{Re}e^{-Q} - \frac{C_r\Omega|\Omega|}{2m}e^Q. \end{cases} \quad (4)$$

As in the case of system (3), there are two singular points with the same properties as in the case of simplified system (3). Denote them by $M_1(Q_1, 0, 0)$ and $M_*(Q_*, 0, 0)$.

Linearisation system (4) gives us only the information on (Q, P) , that is we know that locally $(Q - Q_1, P)$ behaves as follows:

$$(Q - Q_1, P) \approx (a, b)e^{\lambda \xi},$$

where the eigenvector (a, b) and the eigenvalue λ are the same as in the system with $\Omega = 0$. Let us find them. Without loss of generality, we consider the case when $\Omega > 0$. System (4) has the form

$$\begin{cases} Q' = P, \\ P' = \frac{3I}{m^2}e^{-Q} - 3e^{-2Q} - \frac{3g}{2m^2}e^Q - \frac{3\Omega^2}{m^2}e^{2Q} - \frac{12P\Omega}{Rem}e^Q, \\ \Omega' = \frac{4mP^2}{Re}e^{-Q} - \frac{C_r\Omega^2}{2m}e^Q. \end{cases} \quad (5)$$

We linearize system (5) at the point $M_1(Q_1, 0, 0)$. Then the first two equations are separated to give the system

$$\begin{pmatrix} Q' \\ P' \end{pmatrix} = A \begin{pmatrix} Q \\ P \end{pmatrix},$$

where

$$A = \begin{pmatrix} 0 & 1 \\ -\frac{3g}{m^2}e^{Q_1} + 3e^{-2Q_1} & 0 \end{pmatrix}.$$

The characteristic equation of the matrix A is defined by

$$|A - \lambda E| = \begin{vmatrix} -\lambda & 1 \\ -\frac{3g}{m^2}e^{Q_1} + 3e^{-2Q_1} & -\lambda \end{vmatrix} = 0.$$

The eigenvalues are

$$\lambda_1 = \sqrt{-\frac{3g}{m^2}e^{Q_1} + 3e^{-2Q_1}},$$

and

$$\lambda_2 = -\sqrt{-\frac{3g}{m^2}e^{Q_1} + 3e^{-2Q_1}}.$$

Then Proposition 1 implies that

$$-\frac{3g}{m^2}e^{Q_1} + 3e^{-2Q_1} > 0,$$

therefore $\lambda_1 > 0$ and $\lambda_2 < 0$. This corresponds locally to the saddle point: there exist one-dimensional stable and one-dimensional unstable manifolds. The corresponding eigenvectors are

$$\vec{V}_1 = \begin{pmatrix} 1 \\ \lambda_1 \end{pmatrix}, \quad \vec{V}_2 = \begin{pmatrix} 1 \\ \lambda_2 \end{pmatrix}.$$

Now, let us look for Ω in the form: $\Omega \approx \omega e^{\mu t}$, where μ and ω are found as functions of a and $b = a\lambda$. Obviously, in the leading order, we have from the third equation: $\omega \approx \frac{4mb^2}{Re}e^{-Q_1}$ and $\mu = 2\lambda$. Then we can represent the solution in the form

$$\begin{cases} Q - Q_1 & \approx ae^{\lambda_1 \xi}, \\ P & \approx a\lambda_1 e^{\lambda_1 \xi}, \\ \Omega & \approx \frac{4ma^2\lambda_1^2}{Re}e^{-Q_1}e^{2\lambda_1 \xi}, \end{cases} \quad as \quad \xi \rightarrow -\infty.$$

This asymptotic behaviour means that one enters (when ξ is increasing from $-\infty$) at the point $M_1(Q_1, 0, 0)$ into the compact domain shown in Fig. 2 of the surface $L = 0$ along the unstable manifold which is degenerate (Ω is decreasing much more rapidly compared to P , $Q - Q_1$ as $\xi \rightarrow -\infty$). When $\xi \rightarrow \infty$, the solution approaches the point $M_*(Q_*, 0, 0)$ as $\xi \rightarrow +\infty$.

Now consider system (5) at the point $M_*(Q_*; 0; 0)$. Perform the linearization at $M_*(Q_*; 0; 0)$ and get the eigenvalues and eigenvectors in the form:

$$\lambda_1^* = \sqrt{-\frac{3g}{m^2}e^{Q_*} + 3e^{-2Q_*}}, \quad \lambda_2^* = -\sqrt{-\frac{3g}{m^2}e^{Q_*} + 3e^{-2Q_*}}$$

and

$$\vec{V}_1^* = \begin{pmatrix} 1 \\ \lambda_1^* \end{pmatrix}, \quad \vec{V}_2^* = \begin{pmatrix} 1 \\ \lambda_2^* \end{pmatrix}.$$

Proposition 1 implies that

$$-\frac{3g}{m^2}e^{Q_*} + 3e^{-2Q_*} < 0,$$

therefore λ_1^* and λ_2^* are imaginary complex numbers. Since $\lambda_{1,2}^* = ik, k \in R$, the singular point of $M_*(Q_*; 0; 0)$ is a center point considered in the plane of P, Q -variables. However, since L is the Lyapunov function, the solution tends to the state M_* , i.e. the fixed point M_* is asymptotically stable. For a fixed value of C_r , the solution behavior in the neighborhood of the point $M_*(Q_*; 0; 0)$ depends on the Reynolds number. For large Reynolds numbers, it is oscillatory (the case of low dissipation), for small Reynolds numbers it is monotonic (the case of large dissipation). Now we fix the values $C_r = 0,48, h_1 = 1, m = 4, g = 9,8$. The critical value $R = R_c \approx 0,71$ corresponds to the transition from a monotonic profile to an oscillatory one (see Figs. 3 – 5).

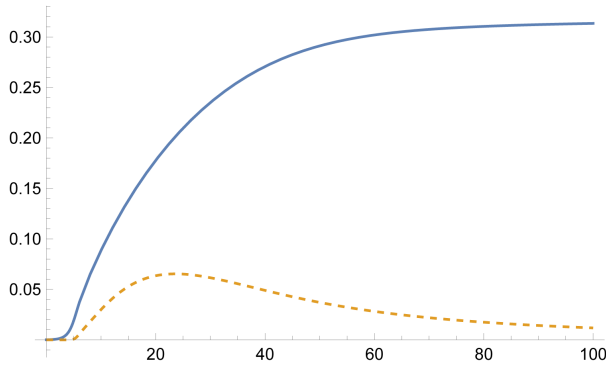


Fig. 3. $R = 0, 1 < R_c \approx 0,71$ and $C_r = 0,48$

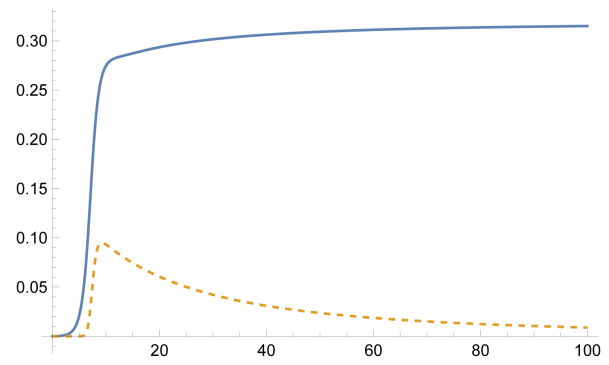


Fig. 4. $R = R_c \approx 0,71$ and $C_r = 0,48$

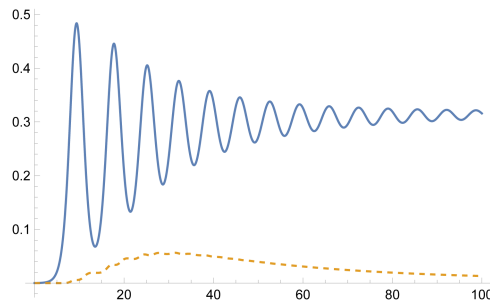


Fig. 5. $R = 10 > R_c$ and $C_r = 0,48$

Conclusion

In this paper, the problem of modelling breaking waves in shallow water is considered. The existence of an analytical and numerical solution to the problem in the form of a traveling wave is shown. The singular points of the system are described. It is shown that there exists a critical value of the Reynolds number corresponding to the transition from a monotonic profile to an oscillatory one.

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ДВИЖУЩИЕСЯ РАЗБИВАЮЩИЕСЯ ВОЛНЫ

Н.М. Кошкарбаев^{1,2}

¹Институт математики и математического моделирования, г. Алматы, Казахстан

²Казахский Национальный Университет им. Аль-Фараби, г. Алматы, Казахстан

Исследуется математическая модель прибрежных волн в приближении мелкой воды. Модель содержит два эмпирических параметра. Первый контролирует турбулент-

ную диссипацию. Второй отвечает за турбулентную вязкость и определяется турбулентным числом Рейнольдса. Мы изучаем решения бегущих волн для этой модели. Показано существование аналитического и численного решения задачи в виде бегущей волны. Описаны особые точки системы. Показано, что существует критическое значение числа Рейнольдса, соответствующее переходу от монотонного профиля к колебательному. Работа организована следующим образом. Во-первых, мы представляем основную систему обыкновенных дифференциальных уравнений (ОДУ) для бегущих волн. Во-вторых, выводится функция Ляпунова для соответствующей системы ОДУ. Наконец, обсуждается поведение решения системы ОДУ.

Ключевые слова: уравнение мелкой воды; функция Ляпунова; число Рейнольдса; решение бегущей волны.

Нурбол Махсетбаевич Кошкарбаев, кафедра математического и компьютерного моделирования, Институт математики и математического моделирования; Казахский национальный университет им. аль-Фараби (г. Алматы, Казахстан), nurbol-koshkarbaev@mail.ru.

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