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AN ANALYSIS OF THE AVALOS–TRIGGIANI PROBLEM FOR THE LINEAR OSKOLKOV SYSTEM OF NON-ZERO ORDER AND A SYSTEM OF WAVE EQUATIONS

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The Avalos–Triggiani problem for a system of wave equations and a linear Oskolkov system of non-zero order is investigated. The mathematical model contains a linear Oskolkov system describing the flow of an incompressible viscoelastic Kelvin–Voigt fluid of non-zero order, and a wave vector equation corresponding to some structure immersed in the fluid. Based on the method proposed by the authors of this problem, the existence of a unique solution to the Avalos–Triggiani problem for the indicated systems is proved.

Keywords: Avalos–Triggiani problem; incompressible viscoelastic fluid; linear Oskolkov system.

Introduction

Let Ω be a bounded domain in $\mathbb{R}^n, n = 2,3$, with sufficiently smooth boundary $\partial\Omega$. Let $u = \mathrm{col}(u_1, u_2, ..., u_n)$ be a $n-$ dimensional velocity vector $n = 2,3$, the scalar function p be a pressure, and the vector $w = \mathrm{col}(w_1.w_2, ..., w_n)$ be a vector of displacement of a body, which occupies the domain Ω_s, and is immersed in a fluid occupying the domain Ω_f. Therefore, $\Omega = \Omega_s \cup \Omega_f, \overline{\Omega}_s \cap \overline{\Omega}_f = \partial\Omega_s \equiv \Gamma_s$ is the common boundary of Ω_s, and Ω_f. Let us denote the outer boundary of Ω_f by Γ_f (see Fig).

Our goal is to investigate the Avalos–Triggiani problem [1, 2] for the case when the fluid in Ω_f is an incompressible viscoelastic Kelvin–Voigt fluid of the nonzero-order [3]. The considered mathematical model is determined by the system

$$(1-\kappa\nabla^2)u_t - \mu\nabla^2 u - \sum_{l=1}^{K}\beta_l\nabla^2\mathbf{w}_l + \nabla p = 0 \qquad \forall(t,x) \in (0,T] \times \Omega_f \equiv \Omega_{Tf}, \qquad (1)$$

$$\frac{\partial \mathbf{w}_l}{\partial t} = u + \alpha_l \mathbf{w}_l, \qquad \alpha_l \in R_-, \qquad \beta_l \in R_+, \qquad l = \overline{1,\ K}, \qquad \forall(t,x) \in \Omega_{Tf}, \qquad (2)$$

$$\nabla \cdot u = 0, \qquad \forall(t,x) \in \Omega_{Tf}, \qquad (3)$$

$$w_{tt} - \nabla^2 w + w = 0 \qquad \forall(t,x) \in (0,T] \times \Omega_s \equiv \Omega_{Ts} \qquad (4)$$

with the boundary value conditions

$$u|_{\Gamma_f} \equiv 0, \qquad \forall(t,x) \in (0,T] \times \Gamma_f \equiv \Gamma_{Tf}, \qquad (5)$$

$$\mathbf{w}_l|_{\Gamma_f} \equiv 0, \qquad \forall(t,x) \in \Gamma_{Tf}, \qquad (6)$$

$$u \equiv w_t, \qquad \forall(t,x) \in (0,T] \times \Gamma_s \equiv \Gamma_{Ts}, \qquad (7)$$

$$\frac{\partial u}{\partial \nu} - \frac{\partial w}{\partial \nu} = p\nu \qquad \forall (t,x) \in \Gamma_{Ts} \tag{8}$$

and the initial value condition

$$(w(0,\cdot), w_t(0,\cdot), \mathbf{w}_1(0,\cdot), \ldots, \mathbf{w}_K(0,\cdot), u(0,\cdot)) = (w_0, w_1, \mathbf{w}_{10}, \ldots, \mathbf{w}_{K0}, u_0) \in \mathbf{H}, \tag{9}$$

where $\mathbf{H} = (H^1(\Omega_s))^n \times (L^2(\Omega_s))^n \times \mathcal{H}_1 \times \ldots \times \mathcal{H}_K \times \mathcal{H}_f$ and $\mathcal{H}_l = (L^2(\Omega_s))^n, l = \overline{1,\ K}$, $\mathcal{H}_f = \{f \in (L^2(\Omega_f))^n : \nabla \cdot f = 0 \text{ in } \Omega_f \text{ and } [f \cdot \nu]|_{\Gamma_f} = 0\}$.

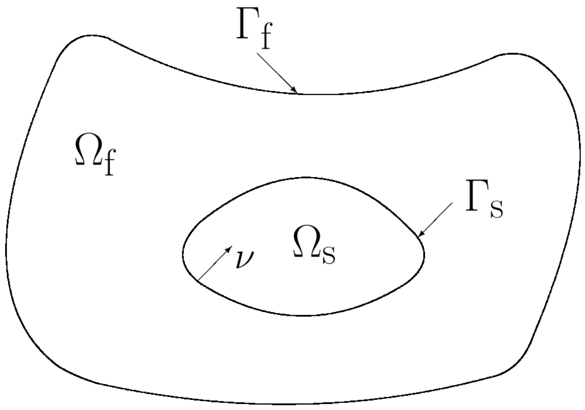

Fig. Physical model

In system (1), the parameters κ and μ characterize the elastic and viscous properties of the fluid, respectively, the parameters β_l, $l = \overline{1,\ K}$ determine the time of pressure retardation (delay), ν is a unit normal vector. In the case of $K = 0, \kappa = 0$, problem (1) – (8) was investigated in [1,2], and for $K = 0, \kappa \neq 0$ in [4], [5]. The case of $K \neq 0, \kappa \neq 0$ is investigated for the first time.

1. Reduction to the Cauchy Problem

Following [1,2], we assume that $p(t)$ satisfies the following elliptic problem:

$$\begin{aligned} &\Delta p = 0 \quad \text{in} \quad \Omega_{Tf}, \\ &p = \frac{\partial u}{\partial \nu} \cdot \nu - \frac{\partial w}{\partial \nu} \cdot \nu \quad \text{on} \quad \Gamma_{Ts}, \\ &\frac{\partial p}{\partial \nu} = \Delta u \cdot \nu \quad \text{on} \quad \Gamma_{Tf}. \end{aligned} \tag{10}$$

Then the pressure p can be represented as follows:

$$p(t) = D_s\{(\frac{\partial u(t)}{\partial \nu} \cdot \nu - \frac{\partial w(t)}{\partial \nu} \cdot \nu)_{\Gamma_{Ts}}\} + N_f((\Delta u(t) \cdot \nu)_{\Gamma_{Tf}}) \quad \text{in} \quad \Omega_{Tf};$$

where the Dirichlet map D_s is defined by the relations

$$h = D_s(g) \Leftrightarrow \begin{cases} \Delta h = 0 \quad \text{in} \quad \Omega_f, \\ h = g \quad \text{on} \quad \Gamma_s, \\ \dfrac{\partial h}{\partial \nu} = 0 \quad \text{on} \quad \Gamma_f, \end{cases}$$

and the Neumann map N_f is defined by the relations

$$h = N_f(g) \Leftrightarrow \begin{cases} \Delta h = 0 \quad \text{in} \quad \Omega_f, \\ h = 0 \quad \text{on} \quad \Gamma_s, \\ \dfrac{\partial h}{\partial \nu} = g \quad \text{on} \quad \Gamma_f. \end{cases}$$

Then original system (1) – (4), which describes the interaction of the fluid and the body immersed in the fluid, takes the form

$$(1-\kappa\nabla^2)u_t - \mu\nabla^2 u - \sum_{l=1}^{K}\beta_l\nabla^2\mathbf{w}_l - G_1 w - G_2 u = 0 \qquad \forall (t,x)\in\Omega_{Tf}, \tag{11}$$

$$\frac{\partial \mathbf{w}_l}{\partial t} = u + \alpha_l \mathbf{w}_l, \qquad \alpha_l \in R_-, \qquad \beta_l \in R_+, \qquad l = \overline{1,\ K}, \tag{12}$$

$$\nabla \cdot u = 0, \tag{13}$$

$$w_{tt} - \nabla^2 w + w = 0 \qquad \forall (t,x)\in\Omega_{Ts} \tag{14}$$

with the boundary value conditions

$$u|_{\Gamma_f} \equiv 0, \qquad \forall (t,x)\in\Gamma_{Tf}, \tag{15}$$

$$\mathbf{w}_l|_{\Gamma_f} \equiv 0, \qquad \forall (t,x)\in\Gamma_{Tf}, \tag{16}$$

$$u \equiv w_t, \qquad \forall (t,x)\in\Gamma_{Ts}, \tag{17}$$

where

$$G_1 w \equiv \nabla\{D_s\{(\frac{\partial w(t)}{\partial \nu}\cdot\nu)_{\Gamma_{Ts}}\}\} \quad \text{in} \quad \Omega_{Tf},$$

$$G_2 u \equiv -\nabla\{D_s\{(\frac{\partial u(t)}{\partial \nu}\cdot\nu)_{\Gamma_{Ts}}\} + N_f((\Delta u(t)\cdot\nu)_{\Gamma_{Tf}})\} \quad \text{in} \quad \Omega_{Tf}.$$

Let us rewrite the problem (11) – (17), in which pressure is excluded, in the form of an abstract Cauchy problem:

$$L\dot v = Mv, \quad v(0) = v_0, \tag{18}$$

where the operators L and M are defined by the matrices

$$L := \begin{pmatrix} I & O & O & O & \dots & O & O \\ O & I & O & O & \dots & O & O \\ O & O & I & O & \dots & O & O \\ O & O & O & I & \dots & O & O \\ \vdots & \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ O & O & O & O & \dots & I & O \\ O & O & O & O & \dots & O & A_\kappa \end{pmatrix},$$

$$M := \begin{pmatrix} O & I & O & O & \dots & O & O \\ \Delta - I & O & O & O & \dots & O & O \\ O & O & \alpha_1 & O & \dots & O & I \\ O & O & O & \alpha_2 & \dots & O & I \\ \vdots & \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ O & O & O & O & \dots & \alpha_K & I \\ G_1 & O & \beta_1\Delta & \beta_2\Delta & \dots & \beta_K\Delta & \nu\Delta + G_2 \end{pmatrix}.$$

Here $v = \mathrm{col}(w, w_t, \mathbf{w}_1, \mathbf{w}_2, \dots, \mathbf{w}_K, u)$, $A_\kappa = 1 - \kappa\nabla^2$, I is a unit operator. Its domain is clear out of the context. We study the problem (18) using the results obtained in [6–9].

Lemma 1. *Let $\kappa \in \mathbb{R}$, $\mu \in \mathbb{R}_+$, the operators L and M be linear continuous operators from $\boldsymbol{G}$ to $\boldsymbol{H}$ ($L, M \in \mathcal{L}(\boldsymbol{G}, \boldsymbol{H})$), then there exists $L^{-1} \in \mathcal{L}(\boldsymbol{H})$. Here the space $\boldsymbol{G} = (H^2(\Omega_s))^n \times (H^2(\Omega_s))^n \times \mathcal{G}_1 \times \ldots \times \mathcal{G}_K \times \mathcal{G}_f$, where $\mathcal{G}_l = (H^2(\Omega_s))^n, l = \overline{1,\ K}$, $\mathcal{G}_f$ is closure according to the norm of the space $(H^2(\Omega_s))^n$ that is the space of infinitely differentiable solenoidal functions such that (15) – (17) are fulfilled.*

Theorem 1. *For any $\kappa \in \mathbb{R}, \mu \in \mathbb{R}_+$ and $v_0 \in \boldsymbol{G}$, there is the unique solution to the problem (18) $v \in C^\infty((0,T], \boldsymbol{G})$*

In conclusion, we note that we intend to develop our research in the direction indicated in [10–12].

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АНАЛИЗ ЗАДАЧИ АВАЛОС – ТРИДЖИАНИ ДЛЯ ЛИНЕЙНОЙ СИСТЕМЫ ОСКОЛКОВА НЕНУЛЕВОГО ПОРЯДКА И СИСТЕМЫ ВОЛНОВЫХ УРАВНЕНИЙ

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В работе исследована задача Авалос – Триджиани для системы волновых уравнений и линейной системы Осколкова ненулевого порядка. Математическая модель содержит линейную систему Осколкова, описывающую течение несжимаемой вязкоупругой жидкости Кельвина – Фойгта ненулевого порядка, и волновое векторное уравнение, соответствующее некоторой структуре, погруженной в указанную жидкость. На основе метода, предложенного авторами задачи, доказана теорема существования единственного решения задачи Авалос – Триджиани для указанных систем.

Ключевые слова: задача Авалос – Триджиани; несжимаемая вязкоупругая жидкость; линейная система Осколкова.

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