КРАТКИЕ СООБЩЕНИЯ

MSC 60H10, 60H30 DOI: 10.14529/mmp240208

ON GLOBAL IN TIME SOLUTIONS OF STOCHASTIC ALGEBRAIC-DIFFERERENTIAL EQUATIONS WITH FORWARD MEAN DERIVATIVES

Y.E. Gliklikh, Voronezh State University, Voronezh, Russian Federation, yeg@math.vsu.ru

The paper is devoted to the investigation of the completeness property of the flows generated by the stochastic algebraic-differential equations given in terms of forward Nelson's mean derivatives. This property means that all solutions of those equations exist for all $t \in [0, \infty)$. It is very important for the description of qualitative behavior of the solutions. This problem is new since previously it was investigated for equations given in terms of symmetric mean derivatives. The case of forward mean derivatives requires different methods of investigation and the cases of forward and symmetric mean derivatives have different important applications. We find conditions under which all solutions of stochastic algebraic-differential equations given in terms of forward Nelson's mean derivatives, exist for all $t \in [0, \infty)$. Some obtained conditions are necessary and sufficient.

Keywords: algebraic-differebtial equations; forvard mean derivatives; global in time solutions.

Introduction

The notion of mean derivatives (forward, backward, symmetric and antisymmetric) was introduced by E. Nelson in [1–3]. In [4] (see also [5] where all preliminaries about mean derivatives are given) an additional mean derivative, called quadratic, was introduced so that from some Nelson's mean derivative and the quadratic one it became in principle possible to find a stochastic process having those derivatives.

In this paper we investigate the completeness property of the flows generated by the stochastic algebraic-differential equations given in terms of forward Nelson's mean derivatives, i.e., we find conditions, under which all solutions of those equations exist for all $t \in [0, \infty)$. Previously, in [6], this problem was investigated for equations given in terms of symmetric mean derivatives. The case of forward mean derivatives requires absolutely different methods of investigation. Some conditions that we have found, are necessary and sufficient.

The structure of the paper is as follows. In Section 1 we give some facts from the theory of matrices, necessary for the description of algebraic-differential equations. Section 2 is devoted to preliminaries of the theory of mean derivatives. In Section 3 we present the main results of the paper.

1. Some Facts from the Theory of Matrices

Everywhere below we deal with the n dimensional linear space $\mathbb{R}^n$, vectors from $\mathbb{R}^n$ and $n \times n$ matrices.

Consider two $n \times n$ constant matrices $\widetilde{L}$ and $\widetilde{M}$ where $\widetilde{L}$ is degenerate while $\widetilde{M}$ is non-degenerate. The expression $\lambda\widetilde{L}+\widetilde{M}$, where λ is real parameter, is called the matrix pencil. The polynomial $\theta(\lambda) = \det(\lambda\widetilde{L} + \widetilde{M})$ is called the characteristic polynomial of the pencil $\lambda\widetilde{L} + \widetilde{M}$. The pencil is called regular, if its characteristic polynomial is not identically equal to zero.

If the matrix pencil $\lambda\widetilde{L} + \widetilde{M}$ is regular, there exist to non-degenerate linear operators P (acts from the left side) and Q (acts from the right side) that reduce the matrices $\widetilde{L}$ and $\widetilde{M}$ to the canonical quasi-diagonal form (see [7]). In the canonical quasi-diagonal form, under appropriate numeration of basis vectors, in the matrix $L = P\widetilde{L}Q$ first along diagonal there is the $d \times d$ unit matrix and then along diagonal there are the Jordan boxes with zeros on diagonal. In $M = P\widetilde{M}Q$ in the lines corresponding to Jordan boxes, there is the unit matrix, and in the lines corresponding to the unit matrix in L there is a certain non-degenerate matrix J. Thus

$$P(\lambda\widetilde{L}(t) + \widetilde{M}(t))Q = \lambda L + M = \lambda \begin{pmatrix} I_d & 0 \\ 0 & N(t) \end{pmatrix} + \begin{pmatrix} J & 0 \\ 0 & I_{n-d} \end{pmatrix}. \qquad (1)$$

Consider a symmetric positive definite (i.e. non-degenerate) $d \times d$ matrix $\Xi(t)$.

Lemma 1. ([4, Lemma 2.2], see also [5]) *There exists a $d \times d$ matrix $A(t)$ such that $\Xi(t) = AA^*$ where A^* is the transposed matrix A.*

2. Mean Derivatives

In this section we briefly describe preliminary facts about mean derivatives. See details in [1–3, 5].

Consider a stochastic process $\xi(t)$ in $\mathbb{R}^n$, $t \in [0, T]$, given on a certain probability space $(\Omega, \mathcal{F}, \mathsf{P})$ and such that $\xi(t)$ is an L_1 random element for all t. It is known that such a process determines 3 families of σ-subalgebras of the σ-algebra $\mathcal{F}$:

(i) “the past” $\mathcal{P}^\xi_t$ generated by preimages of Borel sets from $\mathbb{R}^n$ under all mappings $\xi(s) : \Omega \to \mathbb{R}^n$ for $0 \le s \le t$;

(ii) “the future” $\mathcal{F}^\xi_t$ generated by preimages of Borel sets from $\mathbb{R}^n$ under all mappings $\xi(s) : \Omega \to \mathbb{R}^n$ for $t \le s \le T$;

(iii) “the present” (“now”) $\mathcal{N}^\xi_t$ generated by preimages of Borel sets from $\mathbb{R}^n$ under the mapping $\xi(t) : \Omega \to \mathbb{R}^n$.

All the above families we suppose to be complete, i.e., containing all sets of probability zero.

For the sake of convenience we denote by E^ξ_t the conditional expectation $E(\cdot|\mathcal{N}^\xi_t)$ with respect to the “present” $\mathcal{N}^\xi_t$ for $\xi(t)$.

Following [1–3], introduce the following notions of forward mean derivatives.

Definition 1. *The forward mean derivative $D\xi(t)$ of $\xi(t)$ at the time instant t is an L_1 random element of the form*

$$D\xi(t) = \lim_{\triangle t \to +0} E^\xi_t\left(\frac{\xi(t + \triangle t) - \xi(t)}{\triangle t}\right), \qquad (2)$$

where the limit is supposed to exist in $L_1(\Omega, \mathcal{F}, \mathsf{P})$ and $\triangle t \to +0$ means that $\triangle t$ tends to 0 and $\triangle t > 0$.

One can easily derive that for an Ito process $\xi(t) = \int\limits_0^t a(s)ds + \int\limits_0^t A(s)dw(s)$ its forward mean derivative takes the form $D\xi(t) = a(t)$ since $\int\limits_0^t A(s)dw(s)$ is a martingale and so $D\int\limits_0^t A(s)dw(s) = 0$.

Following [4] (see also [5]) we introduce the differential operator D_2 that differentiates an L_1 random process $\xi(t)$, $t \in [0, T]$ according to the rule

$$D_2\xi(t) = \lim_{\triangle t \to +0} E_t^{\xi}\left(\frac{(\xi(t+\triangle t) - \xi(t))(\xi(t+\triangle t) - \xi(t))^*}{\triangle t}\right), \tag{3}$$

where $(\xi(t+\triangle t)-\xi(t))$ is considered as a column vector (vector in $\mathbb{R}^n$), $(\xi(t+\triangle t)-\xi(t))^*$ is a row vector (transposed, or conjugate vector) and the limit is supposed to exists in $L_1(\Omega, \mathcal{F}, \mathsf{P})$. We emphasize that the matrix product of a column on the left and a row on the right is a matrix. It is shown that $D_2\xi(t)$ takes values in $\bar{S}_+(n)$, the set of symmetric semi-positive definite matrices. We call D_2 the quadratic mean derivative.

One can easily derive that for an Ito process $\xi(t) = \int\limits_0^t a(s)ds + \int\limits_0^t A(s)dw(s)$ its quadratic mean derivative takes the form $D_2\xi(t) = AA^*$ (see [4] and also [5]).

Remark 1. From the properties of conditional expectation (see, e.g., [8]) it follows that there exist Borel mappings $a(t,x)$, and $\alpha(t,x)$ from $R \times \mathbb{R}^n$ to $\mathbb{R}^n$ and to the space of symmetric positive definite matrices, respectively, such that $D\xi(t) = a(t, \xi(t))$ and $D_2\xi(t) = \alpha(t, \xi(t))$. Following [8] we call $a(t,x)$ and $\alpha(t,x)$ the regressions.

3. The Main Result

Let $\Xi(t), t \in [0, \infty)$ be a continuous symmetric positive definite (i.e. non-degenerate) $d \times d$ matrix. By Lemma 1 there exists $d \times d$ matrix A such that $\Xi(t) = A(t)A^*(t)$. Construct the $n \times n$ matrix Θ of the form

$$\Theta = \begin{pmatrix} \Xi(t) & 0 \\ 0 & 0 \end{pmatrix}. \tag{4}$$

We investigate the following stochastic algebraic-differential system

$$\begin{cases} LD\eta(t) = M\eta(t) + f(t), \\ D_2\eta(t) = \Theta, \end{cases} \tag{5}$$

where L and M are from formula (1) and $f(t)$ is a smooth deterministic vector in $\mathbb{R}^n$. Taking into account the struicture of matrices L and M we see that system (5) is decomposed into several independent systems — the one in upper left corner with the unit matrix in L and matrix J in M and the systems corresponding to Jordan boxes in L and the unit matrices in M.

The system in upper left corner takes the form

$$\begin{cases} D\eta_{(1)} = J\eta_{(1)} + f_{(1)}, \\ D_2\eta_{(1)} = \Xi, \end{cases} \tag{6}$$

where $\eta_{(1)}$ and $f_{(1)}$ are constructed from the first d coordinates of the vectors $\eta(t)$ and $f(t)$, respectively.

As an example of the blocs with Jordan matrices in L and the unit matrices in M, we consider $p \times p$ matrix (Jordan box) N in the right bottom corner of L

$$N = \begin{pmatrix} 0 & 1 & 0 & 0 & \dots & 0 \\ 0 & 0 & 1 & 0 & \dots & 0 \\ \vdots & \vdots & \vdots & \vdots & \dots & \vdots \\ 0 & 0 & 0 & 0 & \dots & 1 \\ 0 & 0 & 0 & 0 & \dots & 0 \end{pmatrix},$$

and the corresponding unit matrix from M. The other systems with Jordan boxes are quite analogous. This system takes the form

$$\begin{cases} ND\eta_{(2)}) = \eta_{(2)} + f_{(2)}, \\ D_2\eta_{(2)} = 0, \end{cases} \tag{7}$$

where $\eta_{(2)}$ and $f_{(2)}$ are constructed from the last p coordinates of vectors $\eta(t)$ and $f(t)$, respectively.

Let the greatest Jordan box in L be a $q \times q$ matrix. We investigate the class of systems (5) satisfying the following condition:

Condition. The vector $f(t)$ is uniformly bounded for $t \in [0.\infty)$ together with its derivatives from the first order derivative up to the q-th order derivative.

It is evident that solution of (5) exists for $t \in [0, \infty)$ if and only if the same is valid for solutions of (6) and of (7). We will start with (7).

Theorem 1. *If equation (5) satisfies Condition, the solution of (7) exists for all* $t \in [0, \infty)$.

Proof. First of all, since $D_2\eta_{(2)} = 0$, the process $\eta_{(2)}$ is deterministic and so D turns into ordinary $\frac{d}{dt}$.

In coordinates this system has the form

$$\begin{pmatrix} 0 & 1 & 0 & 0 & \dots & 0 \\ 0 & 0 & 1 & 0 & \dots & 0 \\ \vdots & \vdots & \vdots & \vdots & \dots & \vdots \\ 0 & 0 & 0 & 0 & \dots & 1 \\ 0 & 0 & 0 & 0 & \dots & 0 \end{pmatrix} D \begin{pmatrix} \eta_{(2)}^{n-p} \\ \eta_{(2)}^{n-p+1} \\ \dots \\ \eta_{(2)}^{n-1} \\ \eta_{(2)}^{n} \end{pmatrix} = \begin{pmatrix} \eta_{(2)}^{n-p} \\ \eta_{(2)}^{n-p+1} \\ \dots \\ \eta_{(2)}^{n-1} \\ \eta_{(2)}^{n} \end{pmatrix} + \begin{pmatrix} f_{(2)}^{n-p} \\ f_{(2)}^{n-p+1} \\ \dots \\ f_{(2)}^{n-1} \\ f_{(2)}^{n} \end{pmatrix}. \tag{8}$$

From the last line of (8) we obtain $\eta_2^n = -f_2^n$. From the last but one line of (8) we obtain $\eta_{(2)}^{n-1} = \frac{d}{dt}\eta_{(2)}^n - f_{(2)}^{n-1} = -\frac{d}{dt}f_2^n - f_{(2)}^{n-1}$. Then $\eta_{(2)}^{n-2} = -\frac{d}{dt}\eta_{(2)}^{n-1} - f_{(2)}^{n-2} = -\frac{d^2}{dt^2}f_2^n - \frac{d}{dt}f_{(2)}^p - f_{(2)}^{n-1} - f_{(2)}^{n-2}$, etc. Since Condition is satisfied, all coordinates of $\eta_{(2)}$ exist for $t \in [0, \infty)$.

□

Now we turn to (6). Here we will find several conditions under which the flow, generated by (6), is complete, i.e, the solution of (6) exist for $t \in [0, \infty)$.

Definition 2. *The flow* $\xi(s)$ *is complete on* $[0, T]$ *if every orbit* $\xi_{t,m}(s)$ *a.s. exists for any couple* (t, x) *(with* $0 \le t \le T$*) and for all* $s \in [t, T]$*. The flow* $\xi(s)$ *is complete if it is complete on any interval* $[0, T] \subset \mathbb{R}$.

The structure of equation (6) means that its solution satisfies the following stochastic differential equation in Ito form

$$\eta_{(1)}(t) = \int_0^t J\eta_{(1)}(s)ds + \int_0^t f_{(1)}(s)ds + \int_0^t Adw(s), \tag{9}$$

where A is such that $AA^* = \Xi$ (see above). Thus the generator of the corresponding flow $\xi(s)$ takes the form

$$\mathcal{A}(t,x) = \frac{1}{2}\sum_{i=1 k=1}^{d} \sigma^{ik}\frac{\partial^2}{\partial x^i \partial x^k} + \sum_{i=1}^{d}\sum_{k=1}^{d} j^i_k x^k \frac{\partial}{\partial x^i} + \sum_{k=1}^{d} f^k(t)\frac{\partial}{\partial x^k}, \tag{10}$$

where σ^{ij} are the elements of matrix Ξ, j^i_k are the elements of matrix J and $f^k(t)$ are the coordinates of vector $f(t)$.

Hence the backward equation takes the form

$$\widehat{\eta}(t) = -\int_0^t J\widehat{\eta}_{(1)}(s)ds - \int_0^t f_{(1)}(s)ds + \int_0^t tr\, A'(A)ds - \int_0^t Adw(s) \tag{11}$$

and the backward generator takes the form

$$\widehat{\mathcal{A}}(t,x)f = \frac{1}{2}\sum_{i=1,k=1} \sigma^{ij}\frac{\partial^2}{\partial x^i \partial x^k} - \sum_{i=1}^{d}\sum_{k=1}^{d} j^i_k x^k \frac{\partial}{\partial x^i} - \sum_{k=1}^{d} f^k(t)\frac{\partial}{\partial x^k} + tr\, A'(A). \tag{12}$$

where $tr\, A'(A(t,x))f$ is the derivative of f along the vector field $tr\, A'(A(t,x))$.

Definition 3. *A function from a topological space X to the real line $\mathbb{R}$ is called proper if the preimage of every relatively compact set in $\mathbb{R}$ is relatively compact in X.*

Theorem 2. *Let there exist a smooth proper function φ on $\mathbb{R}^n$ such that $\mathcal{A}(t,x)\varphi < C$ for some $C > 0$ at all $t \in [0,+\infty)$ and $x \in \mathcal{R}^n$ where $\mathcal{A}(t,x)$ is the generator of flow $\xi(s)$. Then the flow $\xi(t,s)$ is complete.*

Theorem 2 is a simple version of rather general sufficient condition [9, Theorem IX. 6A].

Corollary 1. *On $\mathbb{R}\times\mathbb{R}^n$ consider the flow $\widetilde{\xi}(s) = (s, \xi(s))$ with the generator $\widetilde{\mathcal{A}}(t,x) = \frac{\partial}{\partial t} + \mathcal{A}(t,x)$ (see (10)). Let on $\mathbb{R}\times\mathbb{R}^n$ there exist a proper function $\widetilde{\varphi}$ such that $\widetilde{\mathcal{A}}(t,x)\widetilde{\varphi} < C$ for some $C > 0$ at all $t \in [0,+\infty)$ and $x \in \mathcal{R}^n$. Then the flow $\xi(s)$ on $\mathbb{R}^n$ is complete.*

Definition 4. *We say that the flow $\xi(s)$ is continuous at infinity if for any finite interval $[0,T] \subset \mathbb{R}$, any $0 \le t \le T$ and any compact $K \subset M$ the equality*

$$\lim_{x\to\infty} \mathrm{P}(\xi_{t,x}(T)) \in K) = 0 \tag{13}$$

holds where $\xi_{t,x}(s)$ is the orbit of flow $\xi(s)$ such that $\xi_{t,x}(t) = x$.

Theorem 3. ([5, Theorem 7.51], see also [10]) *A flow $\xi(s)$ on $\mathbb{R}^n$ having smooth strictly elliptic generator and being continuous at infinity, is complete on $[0,T]$ if and only if there* *exists a positive proper function* $u^T : \mathbb{R} \times \mathbb{R}^n \to \mathbb{R}$ *that is* C^1*-smooth in* $t \in [0, T]$*,* C^2*-smooth in* $x \in \mathbb{R}^n$ *and such that* $\widetilde{\mathcal{A}} u^T < C$ *for a certain constant* $C > 0$ *at all points* $(t, x) \in \mathbb{R} \times \mathbb{R}^n$ *where* $\widetilde{\mathcal{A}}$ *is the generator of flow* $(s, \xi(s))$ *on* $\mathbb{R} \times \mathbb{R}^n$*.*

Corollary 2. *If the flow* $\xi(s)$ *on* $\mathbb{R}^n$ *with the generator* $\mathcal{A}$ *introduced in* (10) *is continuous at infinity, it is complete if and only if for any* $T > 0$ *there exists a positive proper function* $u^T : \mathbb{R} \times \mathbb{R}^n \to \mathbb{R}$ *that is* C^1*-smooth in* $t \in [0, T]$*,* C^2*-smooth in* $x \in \mathbb{R}^n$ *and such that* $\widetilde{\mathcal{A}} u(t, x) < C$ *for a certain constant* $C > 0$ *at all points* $(t, x) \in \mathbb{R} \times \mathbb{R}^n$*.*

Theorem 4. [11] *Let the backward flow on* $\mathbf{R}^n$ *exist and there exist a smooth positive proper function* u *such that* $\widehat{\mathcal{A}} u < C$ *for a certain constant* $C > 0$*, where* $\widehat{\mathcal{A}}$ *is the generator of backward flow* $\widehat{\xi}(s)$*. Then the forward flow* $\xi(s)$ *is continuous at infinity on* $[0, T]$*.*

Let the flow $\xi(s)$ generated by equation (9) be a flow of diffeomorphisms, i.e., the backward flow exists.

Theorem 5. *The forward flow* $\xi(s)$ *and the backward flow* $\widehat{\xi}(s)$ *generated by equation* (6)*, are simultaneously both complete and continuous at infinity if and only if on* $\mathbb{R} \times \mathbb{R}^n_+$ *there exist positive smooth proper functions* $u(t, x)$ *and* $\widehat{u}(t, x)$ *such that the inequalities*

$$\left(\frac{\partial}{\partial t} + \mathcal{A}\right) u < C \quad \text{and} \quad \left(-\frac{\partial}{\partial t} + \widehat{\mathcal{A}}\right) \widehat{u} < \widehat{C}$$

hold for certain positive constants C *and* $\widehat{C}$*.*

Acknowledgements. *The work is supported by RSCF Grant no. 24-21-00004.*

References

1. Nelson E. Derivation of the Schrödinger Equation from Newtonian Mechanics. *Physic Reviews*, 1966, vol. 150, no. 4, pp. 1079–1085. DOI: 10.1103/PhysRev.150.1079
2. Nelson E. *Dynamical Theory of Brownian Motion*. Princeton, Princeton University Press, 1967. DOI: 10.2307/j.ctv15r57jg
3. Nelson E. *Quantum Fluctuations*. Princeton, Princeton University Press, 1985. DOI: 10.1016/0378-4371(84)90266-8
4. Azarina S.V., Gliklikh Yu.E. Differential Inclusions with Mean Derivatives. *Dynamic Systems and Applications*, 2007, vol. 16, no. 1, pp. 49–71.
5. Gliklikh Yu.E. *Global and Stochastic Analysis with Applications to Mathematical Physics*. London, Springer, 2011.
6. Gliklikh Yu., Sergeeva D. On Conditions for Completeness of Flows Generated by Stochastic Differential-Algebraic Equations. *Global and Stochastic Analysis*, 2021, vol. 8, no. 2, pp. 1–7.
7. Chistyakov V.F., Shcheglova A.A. *Izbrannye Glavy Teorii Algebro-Differencial'Nyh Sistem* [Selected Chapters of the Theory of Algebraic-Differential Systems]. Novosibirsk, Nauka, 2003. (in Russian)
8. Parthasarathy K.R. *Introduction to Probability and Measure*. New York, Springer, 1978. DOI:10.1007/978-1-349-03365-2
9. Elworthy K.D. Stochastic Differential Equations on Manifolds. *Lecture Notes in Statistics*. Cambridge, Cambridge University Press, 1982. DOI: 10.1007/978-1-4612-2224-8_10

10. Gliklikh Yu.E. Necessary and Sufficient Conditions for Global in Time Existence of Solutions of Ordinary, Stochastic, and Parabolic Differential Equations. *Abstract and Applied Analysis*, 2006, vol. 2006, article ID: 39786, 17 p. DOI: 10.1155/AAA/2006/39786
11. Gliklikh Yu.E., Shchichko T.A. On the Completeness of Stochastic Flows Generated by Equations with Current Velocities. *Theory of Probability and Its Applications*, 2019, vol. 64, no. 1, 11 p. DOI: 10.1137/S0040585X97T989350

Received April 9, 2024

УДК 519.216.2 **DOI: 10.14529/mmp240208**

О ГЛОБАЛЬНЫХ ПО ВРЕМЕНИ РЕШЕНИЯХ СТОХАСТИЧЕСКИХ АЛГЕБРО-ДИФФЕРЕНЦИАЛЬНЫХ УРАВНЕНИЙ С ПРОИЗВОДНЫМИ В СРЕДНЕМ СПРАВА

Ю.Е. Гликлих, Воронежский государственный университет, г. Воронеж, Российская Федерация

Статья посвящена исследованию свойства полноты потоков, порожденных стохастическими алгебро-дифференциальными уравнениями, заданными в терминах производных в среднем справа по Нельсону. Это свойство означает, что все решения указанных уравнений существуют при всех $t \in [0, \infty)$. Это важно для описания качественного поведения решений. Это новая задача, поскольку ранее подобная проблема изучалась для уравнений, заданных в терминах симметрических производных в среднем. Случай производных справа требуют других методов исследования и случаи производных справа и симметрических производных имеют разные важные приложения. Мы находим условия, при которых все решения стохастических адгебро-дифференциальных уравнений существуют при $t \in [0, \infty)$. Некоторые из полученных условий являются необходимыми и достаточными.

Ключевые слова: алгебро-дифференциальные уравнения; производные в среднем; глобальные по времени решения.

Исследование поддержано грантом Российского научного фонда № 24-21-00004.

Литература

1. Nelson, E. Derivation of the Schrödinger Equation from Newtonian Mechanics / E. Nelson // Physic Reviews. – 1966. – V. 150, № 4. – P. 1079–1085.
2. Nelson, E. Dynamical Theory of Brownian Motion / E. Nelson. – Princeton: Princeton University Press, 1967.
3. Nelson, E. Quantum Fluctuations / E. Nelson. – Princeton: Princeton University Press, 1985.
4. Azarina, S.V. Differential Inclusions with Mean Derivatives / S.V. Azarina, Yu.E. Gliklikh // Dynamic Systems and Applications. – 2007. – V. 16, № 1. – P. 49–71.
5. Gliklikh, Yu.E. Global and Stochastic Analysis with Applications to Mathematical Physics / Yu.E. Gliklikh. – London: Springer, 2011.
6. Gliklikh, Yu.E. On Conditions for Completeness of Flows Generated by Stochastic Differential-Algebraic Equations / Yu.E. Gliklikh, D. Sergeeva // Global and Stochastic Analysis. – 2021. – V. 8, № 2. – P. 1–7.

7. Чистяков, В.Ф. Избранные главы теории алгебро-дифференциальных систем / В.Ф. Чистяков, А.А. Щеглова. – Новосибирск: Наука, 2003.
8. Партасарати, К. Введение в теорию вероятностей и теорию меры / К. Партасарати. – М.: Мир, 1983
9. Elworthy K.D. Stochastic Differential Equations on Manifolds / K.D. Elworthy // Lecture Notes in Statistics. – Cambridge: Cambridge University Press, 1982.
10. Gliklikh, Yu.E. Necessary and Sufficient Conditions for Global in Time Existence of Solutions of Ordinary, Stochastic, and Parabolic Differential Equations / Yu.E. Gliklikh // Abstract and Applied Analysis. – 2006. – V. 2006. – Article ID: 39786. – 17 p.
11. Gliklikh, Yu.E. On the Completeness of Stochastic Flows Generated by Equations with Current Velocities / Yu.E. Gliklikh, T.A. Shchichko // Theory of Probability and its Applications. – 2019. – V. 64, № 1. – 11 p.

Юрий Евгеньевич Гликлих, доктор физико-математических наук, профессор, профессор кафедры математического и прикладного анализа, Воронежский государственный университет (г. Воронеж, Российская Федерация), yeg@math.vsu.ru.

Поступила в редакцию 9 апреля 2024 г.