MSC 35A01 DOI: 10.14529/mmp240209

# ANALYSIS OF THE AVALOS–TRIGGIANI PROBLEM FOR THE LINEAR OSKOLKOV SYSTEM OF THE HIGHEST ORDER AND A SYSTEM OF WAVE EQUATIONS

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The Avalos–Triggiani problem for a system of wave equations and a linear Oskolkov system of the highest order is investigated. The mathematical model contains a linear Oskolkov system describing the flow of an incompressible viscoelastic Kelvin–Voigt fluid of of the highest order, and a wave vector equation corresponding to some structure immersed in the specified fluid. Based on the method proposed by the authors of this problem, the theorem of the existence of the unique solution to the Avalos–Triggiani problem for the indicated systems is proved.

*Keywords: Avalos–Triggiani problem; incompressible viscoelastic fluid; linear Oskolkov systems.*

## Introduction

Let $\Omega$ be a bounded domain in $\mathbb{R}^n, n = 2, 3$, with sufficiently smooth boundary $\partial\Omega$. Let $u = \mathrm{col}(u_1, u_2, ..., u_n)$ be a $n-$ dimensional velocity vector $n = 2, 3$, the scalar function $p$ be a pressure, and the vector $w = \mathrm{col}(w_1.w_2, ..., w_n)$ be a vector of displacement of a body, which occupies the domain $\Omega_s$, and is immersed in a fluid occupying the domain $\Omega_f$. Therefore, $\Omega = \Omega_s \cup \Omega_f, \overline{\Omega}_s \cap \overline{\Omega}_f = \partial\Omega_s \equiv \Gamma_s$ is the common boundary of $\Omega_s$, and $\Omega_f$. Let us denote the outer boundary of $\Omega_f$ by $\Gamma_f$ (see Fig. 1). Our goal is to investigate the Avalos–Triggiani problem [1, 2] for the case when the fluid in $\Omega_f$ is an incompressible viscoelastic Kelvin–Voigt fluid of the highest order $K(K = n_1 + \ldots + n_M)$ [3]. The considered mathematical model is determined by the system

$$(1 - \varkappa\nabla^2)u_t - \nu\nabla^2 u + (u \cdot \nabla)u - \sum_{m=1}^{M}\sum_{s=0}^{n_m-1} A_{m,s}\nabla^2\mathbf{w}_{m,s} + \nabla p = 0 \tag{1}$$

$$\forall (t, x) \in (0, T] \times \Omega_f \equiv \Omega_{Tf},$$

$$\frac{\partial \mathbf{w}_{m,0}}{\partial t} = u + \alpha_m \mathbf{w}_{m,0},\ \alpha_m \in \mathbb{R}_-,\ m = \overline{1,\ M} \qquad \forall (t, x) \in \Omega_{Tf}, \tag{2}$$

$$\frac{\partial \mathbf{w}_{m,s}}{\partial t} = s\mathbf{w}_{m,s-1} + \alpha_m \mathbf{w}_{m,s},\ s = \overline{1,\ n_m - 1} \qquad \forall (t, x) \in \Omega_{Tf}, \tag{3}$$

$$\nabla \cdot u = 0, \qquad \forall (t, x) \in \Omega_{Tf}, \tag{4}$$

$$w_{tt} - \nabla^2 w + w = 0 \qquad \forall (t, x) \in (0, T] \times \Omega_s \equiv \Omega_{Ts} \tag{5}$$

with the boundary value conditions

$$u|_{\Gamma_f} \equiv 0, \qquad \forall (t,x) \in (0,T] \times \Gamma_f \equiv \Gamma_{Tf}, \tag{6}$$

$$\mathbf{w}_{m,s}|_{\Gamma_f} \equiv 0, \qquad \forall (t,x) \in \Gamma_{Tf}, \tag{7}$$

$$u \equiv w_t, \qquad \forall (t,x) \in (0,T] \times \Gamma_s \equiv \Gamma_{Ts}, \tag{8}$$

$$\frac{\partial u}{\partial \nu} - \frac{\partial w}{\partial \nu} = p\nu \qquad \forall (t,x) \in \Gamma_{Ts} \tag{9}$$

and the initial value condition

$$\begin{aligned}(w(0,\cdot), w_t(0,\cdot), \mathbf{w}_{1,0}(0,\cdot), \ldots, \mathbf{w}_{M,n_m-1}(0,\cdot), u(0,\cdot)) = \\ = (w_0, w_1, \mathbf{w}^0_{1,0}, \ldots, \mathbf{w}^0_{M,n_m-1}) \in \mathbf{H},\end{aligned} \tag{10}$$

where $\mathbf{H} = (H^1(\Omega_s))^n \times (L^2(\Omega_s))^n \times \mathcal{H}_{1,0} \times \ldots \times \mathcal{H}_{M,n_m-1} \times \mathcal{H}_f$ and $\mathcal{H}_{m,s} = (L^2(\Omega_s))^n$, $m = \overline{1,\ M}, s = \overline{1,\ n_m - 1}$, $\mathcal{H}_f = \{f \in (L^2(\Omega_f))^n : \nabla \cdot f = 0 \text{ in } \Omega_f \text{ and } [f \cdot \nu]|_{\Gamma_f} = 0\}$.

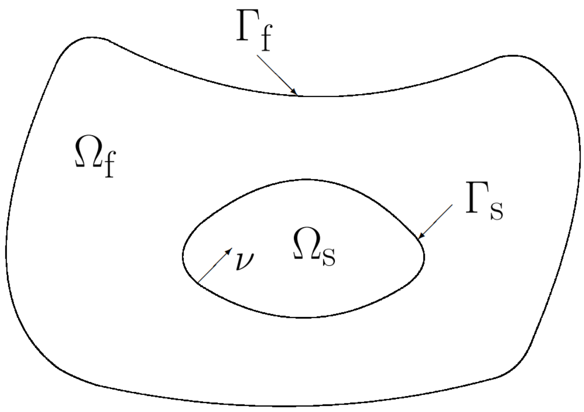


**Fig.** Physical model

In system (1), the parameters $\kappa$ and $\mu$ characterize the elastic and viscous properties of the fluid, respectively, the parameters $A_{m,s}$ determine the time of pressure retardation (delay), $\nu$ is a unit normal vector. In the case of a zero-order Oskolkov system, i.e. $K = 0$, and $\kappa = 0$, problem (1) – (9) was investigated in [1,2], and for $K = 0$ and $\kappa \neq 0$ – in [4,5]. The results of this work generalize the results of [6,7] to the case of the Kelvin–Voigt model of the highest order.

## 1. Reduction to the Cauchy problem

Following [1,2], we assume that $p(t)$ satisfies the following elliptic problem:

$$\begin{aligned}&\Delta p = 0 \quad \text{in} \quad \Omega_{Tf},\\ &p = \frac{\partial u}{\partial \nu} \cdot \nu - \frac{\partial w}{\partial \nu} \cdot \nu \quad \text{on} \quad \Gamma_{Ts},\\ &\frac{\partial p}{\partial \nu} = \Delta u \cdot \nu \quad \text{on} \quad \Gamma_{Tf}.\end{aligned} \tag{11}$$

Then the pressure $p$ can be represented as follows:

$$p(t) = D_s\{(\frac{\partial u(t)}{\partial \nu} \cdot \nu - \frac{\partial w(t)}{\partial \nu} \cdot \nu)_{\Gamma_{Ts}}\} + N_f((\Delta u(t) \cdot \nu)_{\Gamma_{Tf}}) \quad \text{in} \quad \Omega_{Tf};$$

where the Dirichlet map $D_s$ is defined by the relations

$$h = D_s(g) \Leftrightarrow \begin{cases} \Delta h = 0 \quad \text{in} \quad \Omega_f, \\ h = g \quad \text{on} \quad \Gamma_s, \\ \dfrac{\partial h}{\partial \nu} = 0 \quad \text{on} \quad \Gamma_f, \end{cases}$$

and the Neumann map $N_f$ is defined by the relations

$$h = N_f(g) \Leftrightarrow \begin{cases} \Delta h = 0 \quad \text{in} \quad \Omega_f, \\ h = 0 \quad \text{on} \quad \Gamma_s, \\ \dfrac{\partial h}{\partial \nu} = g \quad \text{on} \quad \Gamma_f. \end{cases}$$

Then original system (1) – (5), which describes the interaction of the fluid and the body immersed in the fluid, takes the form

$$(1 - \varkappa\nabla^2)u_t - \nu\nabla^2 u + (u \cdot \nabla)u - \sum_{m=1}^{M}\sum_{s=0}^{n_m-1} A_{m,s}\nabla^2 \mathbf{w}_{m,s} + G_1 w + G_2 u = 0 \tag{12}$$

$$\forall (t,x) \in (0,T] \times \Omega_f \equiv \Omega_{Tf},$$

$$\frac{\partial \mathbf{w}_{m,0}}{\partial t} = u + \alpha_m \mathbf{w}_{m,0},\ \alpha_m \in \mathbb{R}_-,\ A_{m,s} \in \mathbb{R}_+,\ m = \overline{1,\ M} \qquad \forall (t,x) \in \Omega_{Tf}, \tag{13}$$

$$\frac{\partial \mathbf{w}_{m,s}}{\partial t} = s\mathbf{w}_{m,s-1} + \alpha_m \mathbf{w}_{m,s},\ s = \overline{1,\ n_m - 1} \qquad \forall (t,x) \in \Omega_{Tf}, \tag{14}$$

$$\nabla \cdot u = 0, \tag{15}$$

$$w_{tt} - \nabla^2 w + w = 0 \qquad \forall (t,x) \in \Omega_{Ts} \tag{16}$$

with the boundary value conditions

$$u|_{\Gamma_f} \equiv 0, \qquad \forall (t,x) \in \Gamma_{Tf}, \tag{17}$$

$$\mathbf{w}_{m,s}|_{\Gamma_f} \equiv 0, \qquad \forall (t,x) \in \Gamma_{Tf}, \tag{18}$$

$$u \equiv w_t, \qquad \forall (t,x) \in \Gamma_{Ts}, \tag{19}$$

where

$$G_1 w \equiv \nabla\{D_s\{(\frac{\partial w(t)}{\partial \nu} \cdot \nu)_{\Gamma_{Ts}}\}\} \quad \text{in} \quad \Omega_{Tf},$$

$$G_2 u \equiv -\nabla\{D_s\{(\frac{\partial u(t)}{\partial \nu} \cdot \nu)_{\Gamma_{Ts}}\} + N_f((\Delta u(t) \cdot \nu)_{\Gamma_{Tf}})\} \quad \text{in} \quad \Omega_{Tf}.$$

Based on the corresponding results for the operators $L$ and $M$ [8] , problem (12) – (19), in which pressure is excluded, will be written as an abstract Cauchy problem

$$L\dot{v} = Mv, \quad v(0) = v_0, \tag{20}$$

Here $v = \mathrm{col}(w, w_t, \mathbf{w}_{1,0}, \ldots \mathbf{w}_{M,0}, \mathbf{w}_{1,1}, \ldots, \mathbf{w}_{1,l_1} \ldots, \mathbf{w}_{M,1}, \ldots, \mathbf{w}_{M,l_M}, u)$, where $l_m = n_m - 1, m = \overline{1, M}$.

We study the problem (20) using the results obtained in [9–12].

**Lemma 1.** *Let $\kappa \in \mathbb{R}$, $\mu \in \mathbb{R}_+$, the operators $L$ and $M$ be linear continuous operators from $\boldsymbol{G}$ to $\boldsymbol{H}$ ($L, M \in \mathcal{L}(\boldsymbol{G}, \boldsymbol{H})$), then there exists $L^{-1} \in \mathcal{L}(\boldsymbol{H})$. Here is the space $\boldsymbol{G} = (H^2(\Omega_s))^n \times (H^2(\Omega_s))^n \times \mathcal{G}_{1,0} \times \ldots \times \mathcal{G}_{M,n_m-1} \times \mathcal{G}_f$, where $\mathcal{G}_{m,s} = (H^2(\Omega_s))^n$, $m = \overline{1,\ M}$, $s = \overline{1,\ n_m - 1}$, $\mathcal{G}_f$ is closure according to the norm of the space $(H^2(\Omega_s))^n$ of the space of infinitely differentiable solenoid functions such that (17) – (19) are fulfilled.*

**Theorem 1.** *For any $\kappa \in \mathbb{R}, \mu \in \mathbb{R}_+$ and $v_0 \in \boldsymbol{G}$, there is the unique solution to the problem (20) $v \in C^\infty((0,T], \boldsymbol{G})$*

In conclusion, we note that the corresponding stochastic models can also be considered using the approach outlined in [13–15].

**Acknowledgments.** *The work was carried out within the framework of solving problems for the development of the laboratory of Differential Equations and Mathematical Physics of Yaroslav-the-Wise Novgorod State University. The authors express their gratitude to Professor G.A. Sviridyuk for his attention to the work and discussion of the results.*

## References

1. Avalos G., Lasiecka I., Triggiani R. Higher Regularity of a Coupled Parabolic-Hyperbolic Fluid-Structure Interactive System. *Georgian Mathematical Journal*, 2008, vol. 15, no. 3, pp. 403–437. DOI: 10.1515/GMJ.2008.403
2. Avalos G., Triggiani R. Backward Uniqueness of the S.C. Semigroup Arising in Parabolic-Hyperbolic Fluid-Structure Interaction. *Differential Equations*, 2008, vol. 245, no. 3, pp. 737–761. DOI: 10.1016/j.jde.2007.10.036
3. Oskolkov A.P. Initial-Boundary Value Problems for Equations of Motion of Kelvin–Voight Fluids and Oldroyd fluids. *Proceedings of the Steklov Institute of Mathematics*, 1989, vol. 179, pp. 137–182.
4. Sviridyuk G.A., Sukacheva T.G. The Avalos–Triggiani Problem for the Linear Oskolkov System and a System of Wave Equations. *Computational Mathematics and Mathematical Physics*, 2022, vol. 62, no. 3, pp. 427–431.
5. Sukacheva T.G., Sviridyuk G.A. The Avalos–Triggiani Problem for the Linear Oskolkov System and a System of Wave Equaions. II. *Journal of Computational and Engineering Mathematics*, 2022, vol. 9, no. 2, pp. 67–72. DOI: 10.14529/jcem220206
6. Kondyukov A.O., Sukacheva T.G. The Linear Oskolkov System of Nonzero Order in the Avalos–Triggiani Problem. *Journal of Computational and Engineering Mathematics*, 2023, vol. 10, no. 4, pp. 17–23. DOI: 10.14529/jcem230302
7. Sukacheva T.G., Kondyukov A.O. Analysis of the Avalos–Triggiani Problem for the Linear Oskolkov System of Nonzero Order and a System of Wave Equations. *Bulletin of the*

South Ural State University. Series: Mathematical Modelling, Programming and Computer Software*, 2023, vol. 16, no. 4, pp. 93–98. DOI: 10.14529/mmp230407

8. Sukacheva T.G. Unsteady Linearized Model of the Motion of an Incompressible High-Order Viscoelastic Fluid. *Bulletin of the South Ural State University. Series: Mathematical Modelling, Programming and Computer Software*, 2009, no. 17(150), issue 3, pp. 86–93.
9. Oskolkov A.P. Some Nonstationary Linear and Quasilinear Systems Occurring in the Investigation of the Motion of Viscous Fluids. *Journal of Soviet Mathematics*, 1978, vol. 10, pp. 299–335. DOI: 10.1007/BF01566608
10. Sviridyuk G.A., Sukacheva T.G. Phase Spaces of a Class of Operator Semilinear Equations of Sobolev Type. *Differential Equations*, 1990, vol. 26, no. 2, pp. 188–195.
11. Sviridyuk, G.A., Sukacheva, T.G. On the Solvability of a Nonstationary Problem Describing the Dynamics of an Incompressible Viscoelastic Fluid. *Mathematical Notes*, 1998, vol. 63, no. 3, pp. 388–395. DOI: 10.1007/BF02317787
12. Kondyukov A.O., Sukacheva T.G. Phase Space of the Initial–Boundary Value Problem for the Oskolkov System of Nonzero Order. *Computational Mathematics and Mathematical Physics*, 2015, vol. 55, no. 5, pp. 823–828. DOI: 10.1134/S0965542515050127
13. Vasyuchkova K.V., Manakova N.A., Sviridyuk G.A. Some Mathematical Models with a Relatively Bounded Operator and Additive "White Noise". *Bulletin of the South Ural State University. Series: Mathematical Modelling, Programming and Computer Software*, 2017, vol. 10, no. 4, pp. 5–14. DOI: 10.14529/mmp170401
14. Sviridyuk G.A., Zamyshlyaeva A.A., Zagrebina S.A. Multipoint Initial-Final Value for one Class of Sobolev Type Models of Higher Order with Additive "White Noise". *Bulletin of the South Ural State University. Series: Mathematical Modelling, Programming and Computer Software*, 2018, vol. 11, no. 3, pp. 103–117. DOI: 10.14529/mmp180308
15. Favini A., Zagrebina S.A., Sviridyuk G.A. Multipoint Initial-Final Value Problems for Dinamical Sobolev-Type Equations in the Space of Noises. *Electronic Journal of Differential Equations*, 2018, vol. 2018, no. 128, pp. 1–10.

*Received March 5, 2024*

**УДК 517.9** **DOI: 10.14529/mmp240209**

# АНАЛИЗ ЗАДАЧИ АВАЛОС – ТРИДЖИАНИ ДЛЯ ЛИНЕЙНОЙ СИСТЕМЫ ОСКОЛКОВА ВЫСШЕГО ПОРЯДКА И СИСТЕМЫ ВОЛНОВЫХ УРАВНЕНИЙ

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В работе исследована задача Авалос – Триджиани для системы волновых уравнений и линейной системы Осколкова высшего порядка. Математическая модель содержит линейную систему Осколкова, описывающую течение несжимаемой вязкоупругой жидкости Кельвина – Фойгта высшего порядка, и волновое векторное уравнение, соответствующее некоторой структуре, погруженной в указанную жидкость. На основе метода, предложенного авторами задачи, доказана теорема существования единственного решения задачи Авалос – Триджиани для указанных систем.

*Ключевые слова: задача Авалос – Триджиани, несжимаемая вязкоупругая жидкость, линейные системы Осколкова.*

## Литература


1. Avalos, G. Higher Regularity of a Coupled Parabolic-Hyperbolic Fluid-Structure Interactive System / G. Avalos, I. Lasiecka, R. Triggiani // Georgian Mathematical Journal. – 2008. – V. 15, № 3. – P. 403–437.
2. Avalos, G. Backward Uniqueness of the s.c. Semigroup Arising in Parabolic-Hyperbolic FluidStructure Interaction / G. Avalos, R. Triggiani // Differential Equations. – 2008. – V. 245, № 3. – P. 737–761.
3. Осколков, А.П. Начально-краевые задачи для уравнений движения жидкостей Кельвина – Фойгта и Олдройта / А.П. Осколков // Труды Математического института им. В.А. Стеклова. – 1988. – Т. 179. – С. 126–164.
4. Свиридюк, Г.А. Задача Авалос – Триггиани для линейной системы Осколкова и системы волновых уравнений / Г.А. Свиридюк, Т.Г. Сукачева // Журнал вычислительной математики и математической физики. – 2022. – Т. 62, № 3. – С. 437–441.
5. Sukacheva, T.G. The Avalos–Triggiani Problem for the Linear Oskolkov System and a System of Wave Equaions. II / T.G. Sukacheva, G.A. Sviridyuk // Journal of Computational and Engineering Mathematics. – 2022. – V. 9, № 2. – P. 67–72.
6. Kondyukov, A.O. The Linear Oskolkov System of Nonzero Order in the Avalos–Triggiani Problem / A.O. Kondyukov, T.G. Sukacheva // Journal of Computational and Engineering Mathematics. – 2023. – V. 10, № 4. – P. 17–23.
7. Сукачева, Т.Г. Анализ задачи Авалос – Триджиани для линейной системы Осколкова ненулевого порядка и системы волновых уравнений / Т.Г. Сукачева, А.О. Кондюков // Вестник ЮУрГУ. Серия: Математическое моделирование и программирование. – 2023. – Т. 16 № 4. – С. 93–98.
8. Сукачева, Т.Г. Нестационарная линеаризованная модель движения несжимаемой вязкоупругой жидкости высокого порядка / Т.Г. Сукачева // Вестник ЮУрГУ. Серия: Математическое моделирование и программирование. – 2009. – № 17(150), вып. 3. – С. 86–93.
9. Осколков, А.П. О некоторых нестационарных линейных и квазилинейных системах, встречающихся при изучении движения вязких жидкостей / А.П. Осколков // Записки научных семинаров ЛОМИ АН СССР. – 1976. – Т. 59. – С. 133–177.
10. Свиридюк, Г.А. Фазовые пространства одного класса операторных уравнений / Г.А. Свиридюк, Т.Г. Сукачева // Дифференциальные уравнения. – 1990. – Т. 26, № 2. – С. 250–258.
11. Свиридюк, Г.А. О разрешимости нестационарной задачи динамики несжимаемой вязкоупругой жидкости / Г.А. Свиридюк, Т.Г. Сукачева // Математические заметки. – 1998. – Т. 63, № 3. – С. 442–450.
12. Кондюков, А.О. Фазовое пространство начально-краевой задачи для системы Осколкова ненулевого порядка / А.О. Кондюков, Т.Г. Сукачева // Журнал вычислительной математики и математической физики. – 2015. – Т. 55, № 5. – С. 823–829.
13. Васючкова, К.В. Некоторые математические модели с относительно ограниченным оператором и аддитивным «белым шумом» в пространствах последовательностей / К.В. Васючкова, Н.А. Манакова, Г.А. Свиридюк // Вестник ЮУрГУ. Серия: Математическое моделирование и программирование. – 2017. – Т. 10, № 4. – С. 5–14.
14. Свиридюк, Г.А. Многоточечная начально-конечная задача для одного класса моделей соболевского типа высокого порядка с аддитивным «белым шумом» / Г.А. Свиридюк, А.А. Замышляева, С.А. Загребина // Вестник ЮУрГУ. Серия: Математическое моделирование и программирование. – 2018. – Т. 11, № 3. – С. 103–117.

15. Favini, A. Multipoint Initial-Final Value Problems for Dinamical Sobolev-Type Equations in the Space of Noises / A. Favini, S.A. Zagrebina, G.A. Sviridyuk // Electronic Journal of Differential Equations. – 2018. – V. 2018, № 128. – 10 p.

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*Поступила в редакцию 5 марта 2024 г.*