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THE FLUX RECOVERING AT THE ECOSYSTEM-ATMOSPHERE BOUNDARY BY INVERSE MODELLING

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We consider the heat and mass transfer models in the quasistationary case, i. e., all coefficients and the data of the problem depends on time while the time derivative in the equation is absent. Under consideration is the inverse problem of recovering the surface flux through the values of a solution at some collection of points lying inside the domain. The flux is sought in the form of a finite segment of the Fourier series with unknown Fourier coefficients depending on time. The problem of determining the Fourier coefficient is reduced to a system of algebraic equations with the use of special solutions to the adjoint problem. The equation is considered in a cylidrical space domain. We prove the existence and uniqueness theorems for solutions of the corresponding direct problem. The results are employed in the proof of the corresponding results for the inverse problem. The corresponding numerical algorithm in the three-dimensional case is constructed and the results of the numerical experiments are exhibited. It is demonstrated that the algorithm is stable under random perturbations of the data. The finite element method is used. The results can be used in the problem of the determination of the fluxes of green house gases from soils from the concentration measurements.

Keywords: inverse problem; flux; parabolic equation heat and mass transfer.

Introduction

In general, the problem of calculating the dynamics of an admixture in the atmosphere can be mathematically defined as a solution under given initial and boundary conditions of the differential equation [1–4]

$$Mu = \partial u/\partial t + (\vec{a}, \nabla u) = div[K\nabla u] + f,\ K = diag(c_1, c_2, \dots, c_n). \tag{1}$$

Here u is the pollutant concentration minus the background value; $\vec{a}$ is the direction of the wind; the axis x_n is directed vertically upward; $c_i = K_i + D$ $(i = 1, 2, \dots, n)$, with K_i, D the coefficients of turbulent and molecular diffusion (see [5]) and t is time. In view of applications, the equation (1) is often considered in some domain G of the form $G = \Omega \times (0, H)$ (Ω is a bounded domain of the class C^2). Assume that S_0, S_1 are the lower and upper bases of the cylinder G, $\Gamma = \partial G$, $S = \partial\Omega \times (0, H)$. The following initial-boundary conditions are examined: (see [1, 6, 7])

$$u|_S = 0,\ u|_{S_1} = 0,\ c_n\partial u/\partial x_n + \beta u|_{S_0} = g,\ u|_{t=0} = u_0(x). \tag{2}$$

Sometimes, it is reasonable to assume that the flux is given on the lateral surface or on the upper cover of the cylinder G rather than the concentration. In some practical problems, the problem (1), (2) can be simplified. Studying the surface emission of gases, it is possible to observe that the nonstationary summand $\partial C/\partial t$ is essential in some special cases, in particular, in conditions of very weak wind or a low intensity turbulent exchange. The concentration changes are often of quasistationary character and thereby we can exclude the summand $\partial C/\partial t$ equating it to zero and assume only that the coefficients of the equation (1) are known functions of time and space variables [8, p. 19]. The statement of the inverse problem in the general case is as follows. Given the values of concentrations measured at some points $y_i = (y_{1i}, y_{2i}, \ldots, y_{ni})$ $(i = 1, 2, \ldots, r)$, find the function g and a solution C to the problem (1) – (3) such that the given values $\psi_i(t)$ are close to $C(t, y_i)$ or (in the ideal case)

$$u(t, y_i) = \psi_i(t), \quad i = 1, 2, \ldots, r. \tag{3}$$

We look for the function g in the form $g = \sum_{i=1}^{r} q_i(t)\Phi_i(x) + g_0$, where Φ_i is a collection of basis functions, the function g_0 is a given function and the functions q_i are unknowns.

There are two different cases. In the former, the points y_i lie on the boundary of the space domain. In this case the problem is well-posed in the Hadamard sense. In the case of $n = 1$, the uniqueness theorem in this case is established in [9] and the uniqueness and existence theorem of a classical solution in [10] (here the heat flux and the higher-order coefficient depending on time are determined). However, the case of $n = 1$ is rather simple. Probably, the first article devoted to the problem (1) – (3) in the multidimensional case is the article [11] (see also [12]), where, in the case of $Mu = u_t - \Delta u$ and $r = 1$, the existence and uniqueness theorem of classical solutions to the problem was established. More general results were obtained in the article [13]. In the latter case, the points y_i lie in the interior of the domain G. In this case the problem is ill-posed (see some existence and uniqueness results in [14]). At present, there are a series of articles devoted to numerical solving the problems (1) – (3) in different statements and the points $\{y_i\}$ in (3) can be interior [1, 2, 5, 10] or boundary [15, 16]. The main approach is reducing the problem to a control problem and minimization of the corresponding cost functional (see, for instance, [15]). The articles [17–20] are devoted to numerical solving the problem on describing green house gases emission from soils.

Here we examine the quasistationary case, i. e., the equation (1) is replaced with the equation

$$Mu = -div(c(t, x)\nabla u) + \vec{b}(t, x)\nabla u + a(t, x)u = f, \; c = diag(c_1, c_2, \ldots, c_n), \tag{4}$$

and the boundary conditions are of the form

$$c_n u_{x_n}|_{S_0} = g, \; Ru|_S = h, \; u|_{S_1} = g_1, \; g = \sum_{i=1}^{r} q_i(t)\Phi_i(x) + g_0, \tag{5}$$

where $Ru = u$ or $Ru = \partial u/\partial N + \sigma u$. The quasistationary case is considered in [17, 19, 20] and many other articles. The most popular idea of constructing a solution to the inverse problem belongs to Marchuk G.I. [21]. It is also described in [17] and it is based on constructing some particular solutions to the adjoint problem. In the article [20] the question on dependence of a solution $g = g(x, y)$ on the parametrization of the coefficients of the equation is treated, and the function $g \equiv const$ is numerically determined in [19]. In contrast to the other articles, we look for the flux g in the form of a finite segment of the Fourier series. We expose sharp results on the existence and uniqueness of the inverse problem. The corresponding numerical algorithm and the results of the numerical experiments are exhibited in the case of the problem (3) – (5) and $n = 3$. The algorithm is based on the finite element method. It is demonstrated that the problem is stable under random perturbations of the data. The results can be used in the problem of the determination of the fluxes of green house gases from soils (see the statement in [1]).

1. Preliminaries

Let G be a domain in $\mathbb{R}^n$. By $L_p(G)$ and $W_p^s(G)$ $(1 \le p \le \infty)$ we mean the Lebesgue and Sobolev spaces, respectively [22]. Let E be a Banach space. Denote by $L_p(G;E)$ the space of measurable functions defined on G with values in E and the finite norm $\left\| \|u(x)\|_E \right\|_{L_p(G)}$ [22]. We also use the space $C^k(\overline{G};E)$ of E-valued functions continuous in G together with their derivatives up to the order k admitting continuous extensions on the closure $\overline{G}$. The definitions of the Sobolev space $W_p^s(G;E)$ is standard (see [23]).

Proceed with some auxiliary results. Some of them are of interest themselves. Consider the auxiliary problem

$$Lu = -\sum_{i,j=1}^{n} \partial_{x_j}(a_{ij}(t,x)u_{x_i}) + \sum_{i=1}^{n} a_i(t,x)u_{x_i} + a_0(t,x)u + \lambda u = f, \; x \in G, \; t \in (0,T), \quad (6)$$

$$Ru|_\Gamma = h, \; R_i u(t,x',r_i) = g_i(t,x'), \; i = 0,1, \; x' = (x_1,\ldots,x_{n-1}), \quad (7)$$

where $a_{ni} = a_{in} = 0$ for $i = 1,2,\ldots,n-1$, $r_0 = 0, r_1 = H$, $Ru = \sum_{i,j=1}^{n-1} a_{ij}(t,x)\nu_i \frac{\partial u}{\partial x_j} + \sigma(t,x)u$ ($\vec{\nu}$ is the unit outward normal to S) or $Ru = u$, $R_0 u = a_{nn}u_{x_n} + \sigma_0 u$ or $R_0 u = u$, $R_1 u = a_{nn}u_{x_n} + \sigma_1 u$ or $R_1 u = u$, $\lambda \ge 0$, $\lambda \ge 0$ is a real parameter. Describe the conditions on the data. In what follows, we always assume that the operator L is elliptic, i. e., for some constant $\delta_0 > 0$, the inequality $\sum_{i,j=1}^{n} a_{ij}\xi_i\xi_j \ge \delta_0|\xi|^2$ holds for all $x \in G, t \in (0,T)$ and $\xi \in \mathbb{R}^n$. Assume that

$$a_{ij} \in C([0,T]; W_\infty^1(G)), \; a_i \in C([0,T]; L_q(G)) \; (q > n, q \ge p),$$
$$a_0 \in C([0,T]; L_{q_1}(G)), \; (q_1 > n/2, q_1 \ge p), \; i,j = 1,2,\ldots,n; \quad (8)$$

$$f \in C([0,T]; L_p(G)), \; h \in C([0,T]; W_p^{k-1/p}(S)), g_i \in C([0,T]; W_p^{k_i-1/p}(S_i)), \; i = 0,1, \quad (9)$$

where k (or k_i) is equal to 2 if $Ru = u$ (or $R_i u = u$) otherwise, $k = 1$ and, respectively, $k_i = 1$. Moreover, we suppose that if $Ru \ne u$ or (and) $R_i u \ne u$ $(i = 0,1)$ then, respectively,

$$\sigma \in C([0,T]; W_\infty^1(S)), \sigma_i \in C([0,T]; W_\infty^1(S_i)), \; i = 0,1. \quad (10)$$

The consistency conditions are as follows: if $Ru = u$ and $R_i u = u$ $(i = 0,1)$ then $h(t,x',0) = g_i(t,x')|_{\partial\Omega}$; if $Ru = u$ and $R_i u \ne u$ $(i = 0,1)$ then, for $p > 2$, $R_i h(t,x',0) = g_i(t,x')|_{\partial\Omega}$; if $Ru \ne u$ and $R_i u = u$ $(i = 0,1)$ then, for $p > 2$, $R(t,x',r_i)g_i(x')|_{\partial\Omega} = h(t,x',r_i)$. It is possible that the statement of the following theorem is already known. But we did not find direct references and thus the proof is exposed below.

Theorem 1. *Let the conditions* (8) – (10) *and the consistency conditions for every* $t \in (0,T)$ *hold. Assume also that* $p \ne 2$. *Then if a solution to the problem* (6), (7) *is unique in the class* $W_p^2(G)$ *for every* $t \in [0,T]$ *then a solution* u *exists for every* t, $u \in C([0,T]; W_p^2(G))$ *and satisfies the estimate*

$$\|u\|_{C([0,T];W_p^2(G))} \le c(\|f\|_{C([0,T];L_p(G))} + \|h\|_{C([0,T];W_p^{k-\frac{1}{p}}(S))} + \sum_{i=0}^{1} \|g_i\|_{C([0,T];W_p^{k_i-\frac{1}{p}}(S_i))}). \quad (11)$$

If $h = 0, g_1 = 0, g_2 = 0$, and $p \in (1, \infty)$ then there exists a parameter $\lambda_0 \geq 0$ such that for all $\lambda \geq \lambda_0$ there exists a unique solution $u \in C([0,T]; W_p^2(G))$ to the problem (6) – (7) *satisfying the estimate*

$$\|u\|_{C([0,T];W_p^2(G))} + \lambda \|u\|_{C([0,T];L_p(G))} \leq c\|f\|_{C([0,T];L_p(G))}, \tag{12}$$

where the constant c is independent of λ.

Proof. First of all, we note that under the consistency conditions there exists a function $\Phi \in C([0,T]; W_p^2(G))$ satisfying the boundary conditions (7) (see Theorem 7.3 in [25]). After the change of variables $u = v + \Phi$, we arrive at the problem

$$Lv = f, \ Ru|_S = 0, \ R_i u(t, x', r_i) = 0, \ i = 0, 1, \tag{13}$$

where the same the notation for the new right-hand side is employed. Without loss of generality, we can assume that if $R_i u \neq u$ then $\sigma_i = 0$. Since the summands $\sigma_i u$ in the boundary condition are lower-order terms, the case of $\sigma_i \not\equiv 0$ can be easily considered with the help of the method of continuation in a parameter, for example. Consider the case of the boundary conditions $R_0 u = u_{x_n}$, $R_1 u = u_{x_n}$ $Ru = u$. The remaining cases are treated by analogy. Construct a function $\psi_0(x_n) \in C^\infty(\mathbb{R})$ even in the variable $x_n \in (-H, H)$ and such that $supp\, \psi_0 \in (-2H/3, 2H/3)$, $\psi_0 = 1$ for $x_n \in [0, H/2]$. Define also a function $\psi_1(x_n)$ even with respect to the point $x_n = H$ and such that $\psi_1(x_n) = 1 - \psi_0(x_n)$ for $x_n \in (0, H)$ and $\psi_1(x_n) = 0$ for $x_n < 0$. Construct also functions $\alpha_i(x_n) \in C^\infty(\mathbb{R})$ with the same properties such that $supp\, \alpha_0 \subset (-2H/3, 2H/3)$, $supp\, \alpha_1 \subset (H/3, 5H/3)$, $\alpha_i = 1$ on $supp\, \psi_i$ $(i = 0, 1)$ and $\alpha_0 = 1$ on $supp\, \psi_1$. Construct domains G_0 and G_1 such that $G_0 \supset \Omega \times [0, 2H/3]$, $G_0 \subset \Omega \times [0, 3H/4)$, $G_1 \supset \Omega \times [H/3, H]$, $G_1 \subset \Omega \times [H/4, H]$, and the parts of the boundaries ∂G_0 and ∂G_1 lying in the domains $x_n > 0$ and $x_n < H$, respectively, belong to the class C^2. Next, we construct the extensions of G_0 and G_1 symmetric with respect to the planes $x_n = 0$ and $x_n = H$. Denote these extensions by the same symbols. By construction, $\partial G_0 \in C^2$ and $\partial G_1 \in C^2$. Given a function $\varphi \in L_p(G)$, extend it to $Q_H = \Omega \times (-H, 2H)$ taking $\tilde{\varphi}(t, x', x_n) = \varphi(t, x', x_n)$ for $x_n \in (0, H)$, $\tilde{\varphi} = \varphi(t, x', -x_n)$ for $x_n \in (-H, 0)$, $\tilde{\varphi}(t, x', x_n) = g(x', 2H - x_n)$ for $x_n \in (H, 2H)$. We have $\tilde{\varphi} \in L_p(Q_H)$. Extend the coefficients $a_{ij}, a_i (i \neq n)$ as even functions and a_n as an odd function with respect to the points $x_n = 0$ and $x_n = H$ into Q_H. Consider the problems

$$Lu_i = \tilde{g}\alpha_i, \quad x \in G_i, \ i = 0, 1, \tag{14}$$

$$u_i|_{\partial G_i} = 0, \ i = 0, 1. \tag{15}$$

By Theorem 8.2 [24], there exists $\lambda_0 \geq 0$ such that the problems (14), (15) are uniquely solvable for $\lambda \geq \lambda_0$ and the estimate

$$\|u_i\|_{W_p^2(G_i)} + \lambda \|u_i\|_{L_p(G_i)} \leq c\|g\|_{L_p(G_i)}, \ i = 0, 1 \tag{16}$$

holds. Due compactness of the segment $[0, T]$, we can assume that the constant c is independent of time. In our case G_0 is symmetric with respect to the planes $x_n = 0$ and a solution u_0 possesses the property $u_{0x_n}|_{x_n=0} = 0$. Indeed, consider the function $\tilde{u}_0(t, x) = u_0(t, x', -x_n)$ for $x \in G_0$. In this case $\tilde{u}_0|_{\partial G_0} = 0$, $\tilde{u}_{0x_n} = -u_{0x_n}(t, x', -x_n)$, $\tilde{u}_{0x_i} = u_{0x_i}(t, x', -x_n)$, $\tilde{u}_{0x_n x_n} = u_{0x_n x_n}(t, x', -x_n)$, $\tilde{u}_{0x_i x_j} = u_{0x_i x_j}(t, x', -x_n)$ for $i, j \leq n-1$.

This function $\tilde{u}_0$ is also a solution to the problem (14), (15). The uniqueness theorem implies that $\tilde{u}_0 = u_0$. Thus, the function u_0 is even in x_n and thereby $u_{0x_n}(t, x', 0) = 0$. Similarly, $u_{1x_n}(t, x', H) = 0$ and u_1 is an even function in the variable x_n with respect to the point $x_n = H$. A solution to our problem is sought in the form $v = u_0\psi_0 + u_1\psi_1$. The function v meets the boundary conditions in G, i. e. the Dirichlet condition on $\partial\Omega \times (0, H)$ and the Neumann condition on the planes $x_n = 0$ and $x_n = H$. Inserting v in the equation, we have $Lv = \psi_0 Lu_0 + [L, \psi_0]u_0 + \psi_1 Lu_1 + [L, \psi_1]u_1 = g + [L, \psi_0]u_0 + [L, \psi_1]u_1$, where $[L, \psi_i]b = L(\psi_i b) - \psi_i Lb$. The function g must satisfy the equation

$$g + V(g) = f, \quad V(g) = [L, \psi_0]u_0 + [L, \psi_1]u_1. \tag{17}$$

Estimate the norm $\|V(g)\|_{L_p(G)}$. By definition, $[L, \psi_k]u_k = L(\psi_k u_k) - \psi_k Lu_k = -2a_{nn}\psi_{kx_n}u_{kx_n} - a_{nn}u_k\psi_{kx_nx_n} + a_n\psi_{kx_n}u_k$ $(k = 0, 1)$. We have the embeddings $W_p^1(G) \subset L_{q_0}(G)$ ($q_0 \le np(n-p)$ for $p < n$ and arbitrary for $p = n$), $W_p^1(G) \subset L_\infty(G)$ for $p > n$ [22]. Let, for example, $p < n$. In this case (see (8))

$$\|V(g)\|_{L_p(G)} \le \sum_{k=0}^{1} c\|u_{kx_n}\|_{L_p(G_k)} + c_1\|a_n\|_{L_q(G)}\|u_k\|_{L_{pq/(q-p)}(G_k)} \le c_2 \sum_{k=0}^{1} \|u_k\|_{W_p^1(G_k)}. \tag{18}$$

The case $p \ge n$ is treated by analogy. The interpolation inequality $\|u\|_{W_p^1(G_k)} \le c\|u\|^{1/2}_{W_p^2(G_k)}\|u\|^{1/2}_{L_p(G_k)}$ and the estimate (16) yield

$$\|V(g)\|_{L_p(G)} \le c_3|\lambda|^{-1/2}\|g\|_{L_p(G)}, \ t \in [0, T], \tag{19}$$

where the constant c_2 is independent of $\lambda \ge \lambda_0$ and $t \in [0, T]$. Fix $q \in (0, 1)$. Increasing the parameter λ_0 if necessary, we can assume that $c_3|\lambda|^{-1/2} \le q$. In this case the equation (17) is uniquely solvable and a solution satisfies the estimate $\|g\|_{C([0,T];L_p(G))} \le c_4\|f\|_{C([0,T];L_p(G))}$, $c_4 = 1/(1-q)$. Moreover, the function $v = u_1\psi_1 + u_2\psi_2$ is a solution to the initial problem satisfying the estimate (12). The first part of the theorem results from the Fredholm theory.

□

Next, we discuss the generalised solvability of the problem (6), (7) in the case of $g, g_i \equiv 0$. Denote by $W_{p,B}^1(G)$ the space of functions in $W_p^1(G)$ satisfying those boundary conditions in (7) that have a sense. Therefore, if $Ru \ne u$, $R_0u \ne u$, and $R_1u \ne u$ then $W_{p,B}^1(G) = W_p^1(G)$. Denote by $W_{q,B^*}^{-1}(G)$ $1/p + 1/q = 1)$ the dual space to $W_{p,B}^1(G)$ with respect to the duality pairing defined by the inner product in $L_2(G)$. The adjoint problem to the problem (6), (7) with the homogeneous data is written as:

$$L^*v = f, \ \tilde{R}v|_S = 0, \ \tilde{R}_iv(x', r_i) = 0, \ i = 0, 1, \tag{20}$$

where $\tilde{R}v = Rv + \vec{a} \cdot \vec{\nu}v$ if $Rv \ne v$ and $\tilde{R}v = v$ if $Rv = v$, $\tilde{R}_0v = R_0v + a_nv$ if $R_0v \ne v$ and $\tilde{R}_0v = v$ if $R_0v = v$, $\tilde{R}_1v = R_1v + a_nv$ if $R_1v \ne v$ and $\tilde{R}_1v = v$ if $R_1v = v$. Here $\vec{a} \cdot \vec{\nu} = \sum_{i=1}^n a_i\nu_i$.

Denote by $W_{p,B}^{-1}(G)$ $(1/p + 1/q = 1)$ the dual to $W_{q,B^*}^1(G)$ with respect to the duality pairing defined by the inner product in $L_2(G)$, where $W_{q,B^*}^1(G)$ is the space of functions in $W_q^1(G)$ satisfying those boundary conditions in (20) that have a sense. The condition on the data are as follows:

$$a_{ij} \in C([0, T]; W_\infty^1(G)), \ \sigma \in C([0, T]; W_\infty^1(S)), \ \sigma_i(x') \in C([0, T]; W_\infty^1(S_i)), \ i = 0, 1, \tag{21}$$

where the last inclusions are fulfilled when $Ru \neq u$ or $R_i u \neq u$ for some $i = 0, 1$, respectively;

$$a_0 \in C([0,T]; L_{q_1}(G)), a_i \in C([0,T]; W^1_{q_1}(G)) \ (q_1 > \max(n,p,q)), \ i = 1, \ldots, n, \tag{22}$$

where $q = p/(p-1)$, $p \in (1, \infty)$. Define a generalized solution to the problem (6), (7) in the case of $h, g_i \equiv 0$. Let, for instance, $Ru \neq u$ and $R_i u \neq u$ $(i = 0, 1)$. By a generalized solution to the problem, we mean a function $u \in C([0,T]; W^1_{p,B}(G))$ such that

$$\int\limits_G \sum_{i,j=1}^n a_{ij} u_{x_i} \varphi_{x_j} + (\sum_{i=1}^n a_i u_{x_i} + a_0 u)\varphi \, dx + \int\limits_S \sigma u \varphi \, dS - \int\limits_{S_0} \sigma_0 u \varphi \, dx' + \int\limits_{S_1} \sigma u \varphi \, dx' =$$
$$= \int\limits_G f(t,x)\varphi(t,x) \, dx, \ \forall \varphi \in C([0,T]; W^1_{q,B^*}(G)), \ t \in [0,T].$$

Similar definitions are introduced in the remaining cases.

Theorem 2. *Let the conditions* (21), (22) *hold and let* $f \in C([0,T]; W^{-1}_{p,B}(G))$ *and* $h, g_i \equiv 0$. *Then if a solution to the problem* (6), (7) *is unique in the class* $W^1_{p,B}(G)$ *for every* $t \in [0,T]$ *then a solution* u *to this problem exists for every* t *and* $u \in C([0,T]; W^1_{p,B}(G))$. *There exists a parameter* $\lambda_0 \geq 0$ *such that, for all* $\lambda \geq \lambda_0$, *there exists a unique solution* $u \in C([0,T]; W^1_{p,B}(G))$ *to the problem* (6), (7) *satisfying the estimate* $\|u\|_{C([0,T];W^1_p(G))} + \lambda \|u\|_{C([0,T];W^{-1}_{p,B}(G))} \leq c\|f\|_{W^{-1}_{p,B}(G)}$, *where the constant* c *is independent of* λ.

Proof. The proof is carried out along the same lines as those in the previous theorem. Instead of the results in [24], we refer to Theorems 12.2, 14.2 [26]. □

Corollary 1. *Let* $f = \sum_{i=1}^r q_i(t)\delta(x - y_i)$ $(y_i \in G$, *and* δ *is the Dirac delta-function) and* $q_i(t) \in C([0,T])$. *Assume that the condition* (22) *holds and*

$$a_{ij} \in C([0,T]; W^1_\infty(G)), \ \sigma + \vec{a}\cdot\vec{\nu} \in C([0,T]; W^1_\infty(S)), \ \sigma_i(x') + a_n \in C([0,T]; W^1_\infty(S_i)), \tag{23}$$

where the last inclusions are fulfilled when $Ru \neq u$ *or* $R_i u \neq u$ *for some* $i = 0, 1$, *respectively. Then* $f \in C([0,T]; W^{-1}_{q,B^*}(G))$ *with* $q \in [1, n/(n-1))$, *where* $q = p/(p-1)$, $p \in (1, \infty)$, *and if a solution to the problem* (20) *is unique in the class* $W^1_{q,B^*}(G)$ *for every* $t \in [0,T]$ *then there exists a unique solution* v *to the problem* (20) *such that* $v \in C([0,T]; W^1_{q,B^*}(G))$.

Remark 1. It is sometimes possible to take Ω with a Lipschitz boundary. For example, if $n = 3$ and $\Omega = (a,b) \times (c,d)$ then all statement of Theorems 1-2 are valid whenever the operator L agrees with the operator M in (4). We only must take into account that additional consistence conditions can appear at the points $x_1 = a, b, x_2 = c, d$. The remaining statement are of the same form.

2. Existence and Uniqueness Theorems

Under consideration is the inverse problem

$$Lu = -\sum_{i,j=1}^n \partial_{x_j}(a_{ij}(t,x)u_{x_i}) + \sum_{i=1}^n a_i(t,x)u_{x_i} + a_0(t,x)u + \lambda u = f, \tag{24}$$

$$Ru|_\Gamma = h,\ R_1u(t,x',H) = g_1(t,x'),\ R_0u(t,x',0) = g(t,x'),\ u(t,y_i) = \psi_i(t), \tag{25}$$

where $i = 1,\dots r$, $R_0u = a_{nn}u_{x_n} + \sigma_0 u$, and $g = \sum_{i=1}^r q_i(t)\Phi_i(x') + g_0$ (Φ_i is a collection of linearly independent functions on Ω and the functions $q_i(t)$ are unknowns). The adjoint problem to the problem (6), (7) is written as

$$L^*v = f,\ \tilde{R}v|_S = 0,\ \tilde{R}_iv(t,x',r_i) = 0,\ i = 0,1. \tag{26}$$

Fix $p > n/2$, $p \neq 2$. Next, let the conditions (9), (10), (22), (23) hold. Assume also that $\Phi_j \in W_p^{1-1/p}(\Omega)$ for all j. If $Ru = u$ and $p > 2$ then we suppose that $\Phi_i(x')|_{\partial\Omega} = 0$, $i = 1,2,\dots,r$. The consistency conditions are written as follows: for every $t \in [0,T]$, if $Ru = u$ and $R_1u = u$ then $h(t,x',H) = g_1(t,x')|_{\partial\Omega}$; if $Ru = u$, $R_iu \neq u$ $(i = 0,1)$ and $p > 2$ then $R_ih(t,x',0) = g_i(t,x')|_{\partial\Omega}$ $(i = 0,1)$; if $Ru \neq u$, $R_1u = u$, and $p > 2$ then $R(t,x',r_i)g_1(t,x')|_{\partial\Omega} = h(t,x',r_1)$. Under the above conditions, there exists a function $\Phi \in C([0,T];W_p^2(G))$ such that $R\Phi|_S = h$, $R_1\Phi(x',r_1) = g_1$, $R_0\Phi = g_0$ [25, Theorem 7.3]. Making the change of variables $u = v + \Phi$, we arrive at the problem

$$Lv = -\sum_{i,j=1}^n \partial_{x_j}(a_{ij}(t,x)v_{x_i}) + \sum_{i=1}^n a_i(t,x)v_{x_i} + a_0(t,x)v + \lambda v = f,\ x \in G, \tag{27}$$

$$Rv|_\Gamma = 0,\ R_1v(t,x',r_i) = 0,\ R_0v(t,x',0) = \sum_{i=1}^r q_i(t)\Phi_i(x'), \tag{28}$$

$$v(t,y_j) = \psi_j(t) - \Phi(t,y_j) = \tilde{\psi}_j,\ j = 1,2,\dots,r. \tag{29}$$

Let $v \in C([0,T];W_p^2(G))$ be a solution to this problem. The condition $p > n/2$ ensures the embedding $W_p^2(G) \subset W_{q_0}^1(G)$, with $q_0 > n$. In this case $q_0' \in (1, n/(n-1))$, with $q_0' = q_0/(q_0-1)$. If the conditions of Corollary 1 are fulfilled then there exist solutions $v_j \in C([0,T];W^1_{q_0',B^*}(G))$ to the problems (26), with $f = \delta(x-y_j) \in C([0,T];W^{-1}_{q_0',B^*}(G))$. It is not difficult to justify multiplying (27) by v_j and integrating by parts that

$$\int_\Omega R_0v(t,x',0)v_j(t,x',0)\,dx' + \tilde{\psi}_j(t) = \int_G fv_j\,dx,$$

where the right-hand side is the value of the functional f on the function v_j. The boundary condition (28) leads to the system

$$\sum_{i=1}^r q_i \int_\Omega \Phi_i(x')v_j(t,x',0)\,dx' = \int_G fv_j\,dx - \tilde{\psi}_j(t),\ j = 1,2,\dots,r, \tag{30}$$

which can be written in the form

$$A\vec{\alpha} = \vec{F},\ F_i = \int_G fv_i\,dx - \tilde{\psi}_i(t),\ a_{ij} = \int_\Omega \Phi_jv_i(t,x',0)\,dx'. \tag{31}$$

Assume that

$$\det A \neq 0\ \forall t \in [0,T],\ \psi_i(t) \in C([0,T])\ (i = 1,2,\dots,r). \tag{32}$$

The matrix A has the entries $a_{ij} = \int\limits_{\Omega} \Phi_j v_i(t,x',0)\,dx'$. Thus, to solve the inverse problem, we need to determine the functions v_j and to solve the system (31).

Theorem 3. *Assume that the conditions* (8) – (10), (22), (23), (32) *and the consistency conditions hold, $p > n/2$, and a solution to the problem* (6),(7) *from the class* $W_p^2(G)$ *is unique for every* $t \in [0,T]$. *Then there exists a unique solution* $u, q_1, q_2, \ldots, q_r$ *to the problem* (24),(25) *such that* $u \in C([0,T]; W_p^2(G))$, $q_i \in C([0,T])$ $(i = 1,\ldots,r)$. *If the conditions of the theorem hold and there exists a segment* $[t_1,t_2]$ $(t_1 < t_2)$ *such that* $\det A = 0$ $\forall t \in [t_1,t_2]$ *then a solution to the problem* (24),(25) *is not unique in the class* $C([0,T]; W_p^2(G))$.

Proof. First of all, we note that the duality arguments allow to show that if a solution to the problem (6),(7) from the class $W_p^2(G)$ is unique for every $t \in [0,T]$ then a solution to the problem (26) from the class $W^1_{q,B^*}(G)$ is unique for every t as well. In this case we can construct the functions v_j $(j = 1,2,\ldots,r)$ to the problem (26) with $f = \delta(x - y_j)$ and the former part of the theorem results from the fact that the system (31) is uniquely solvable. Next, we recover the function v as a solution to the problem (27), (28), and determine the function $u = v + \Phi$. Demonstrate that a solution v to the problem (27), (28) meets (29). The definition of a generalized solution to the problem (27), (28) and the properties of the functions v_j imply the equalities

$$\int\limits_{\Omega} (a_{nn} v_{x_n}(t,x',0) + \sigma_0 v(t,x',0)) v_j(t,x',0)\,dx' + v(t,y_j) = \int\limits_{G} f v_j\,dx,$$

subtracting which from the equalities (30), we justify (29).

Now, assume that $\det A = 0$ $\forall t \in [t_1,t_2]$. Take $t^0 \in (t_1,t_2)$ Let $r(A(t_0)) = \beta < r$. In this case, either there exists a neighborhood about t_0 in which $r(A(t)) = \beta < r$ (the rank of $A(t)$) or in any neighborhood there exists a point t^1 such that $\beta < r(A(t^1)) < r$. In the latter case, choose $t^1 \in (t_1,t_2)$ such that $\beta < r(A(t^1)) < r$. Repeating the arguments finitely many times, we can find a point $t^k \in (t_1,t_2)$ with a neighborhood $U \subset (t_1,t_2)$ such that $r(A(t)) = \beta_0 < r$ in U. Without loss of generality, we can assume that the matrix lying at the first β_0 columns and rows has the rank β_0 and its determinant does not vanish in U. Instead of the system (31) with the matrix $A = \{\alpha_{ij}\}$, we take the system

$$\sum_{j=1}^{\beta_0} \alpha_{ij} q_j = -\sum_{j=\beta_0+1}^{r} \alpha_{ij} q_j.$$

Take arbitrary functions q_j $(j = \beta_0+1,\ldots,r)$ of the class $C_0^\infty(U)$. The remaining functions $\{q_j\}$ are solution to this system. These functions belongs to the class $C(\overline{U})$ and are compactly supported in U. Extending them by zero on the segment $[0,T]$ and solving the direct problem (27), (28) with these functions we determine a nontrivial solution to the homogeneous problem.

□

3. Numerical Algorithm

We take $n = 3$ and $G = \Omega \times (0, Z)$. We examine the problem (3) – (5), where the boundary conditions are of the form

$$c_3 u_{x_3}|_{S_0} = g(t, x),\ u|_{S_1} = 0,\ u|_S = 0,\ g = \sum_{i=1}^{r} q_i(t)\Phi_i(x'). \tag{33}$$

Let $y_m = (y_{1m}, y_{2m}, y_{3m})$. The finite element method is employed. Divide the domain G into tetrahedra and construct the corresponding piecewise linear basis $\{\varphi_i(x)\}_{i=1}^N$. The nodes of the grid are denoted by $\{p_i\}_{i=1}^N$. For convenience, we assume that the points y_m are the nodes p_{i_m} (m=1,2,. . . ,r) of the grid. An approximate solution has the form $v = \sum_{i=1}^N C_i(t)\varphi_i(x)$ and the functions $C_k(t)$ are solutions to the system

$$\sum_{k=1}^{N} a_{ik}C_k = \vec{F},\ a_{ik} = \int_G \sum_{j=1}^{3}(c_j(t,x)\varphi_{kx_j}\varphi_{ix_j} + b_j(t,x)\varphi_{kx_j}\varphi_i) + a\varphi_k\varphi_i\,dx, \tag{34}$$

where $\vec{F} = \vec{F}_0 - \sum_{j=1}^r q_j(t)\vec{F}_j$ and $\vec{F}_j = (\int_\Omega \Phi_j\varphi_1(x',0)\,dx', \ldots, \int_\Omega \Phi_j\varphi_N(x',0)\,dx')^T$, $\vec{F}_0 = ((f,\varphi_1), \ldots, (f,\varphi_N))^T$. Thus, the system can be written in the matrix from

$$A\vec{C} = \vec{F}. \tag{35}$$

Let e_j be a basis vector whose jth coordinate is equal to 1 and the remaining coordinates vanish. Define the vector $\vec{v}_j = (A^*)^{-1}e_{i_j}$ $(j = 1, 2, \ldots, r)$, where A^* is the transposed matrix. Find the quantities q_i. To this end, multiply the equation (35) scalarly by $\vec{v}_j$ and use the conditions (3) which in our case have the from $C_{i_j} = \psi_j$. We derive that

$$< \vec{F}_0, \vec{v}_j > -\psi_j = \sum_{i=1}^{r} q_i < \vec{F}_i, \vec{v}_j >,\ j = 1, 2, \ldots, r,$$

where the brackets $< \cdot,\cdot >$ stand for the inner product in $\mathbb{R}^N$. The vectors $\vec{q}$ can be determined from this system. The vector $\vec{C}$ is a solution to the system (35).

The matrix with entries $< \vec{F}_i, \vec{v}_j >$ is a discrete analog of the matrix A in (32). In the case of a regular family of finite elements, it is possible to prove the convergence of the entries $< \vec{F}_j, \vec{v}_i >$ to the corresponding entries of the matrix A (see Theorems 3.1.5, 3.1.6 [27]). This means that if the condition (32) is fulfilled then the determinant of A in (34) also does not vanish for sufficiently small partition of the domain.

4. Numerical Realization

In this section we present the results of numerical experiments for some collections of the data. To determine the accuracy of calculations, we take given functions c, $\vec{b}$, a, f and the function g depending on the known functions Φ_i and q_i (see (5)). Solving the direct problem (4), (5) we can find a solution u to this problem and thereby the quantities $u(t, y_i) = \psi_i$ $(i = 1, 2, \ldots, r)$. Next, using this data we can solve the inverse problem (3) – (5), and find a solution u and the functions q_i. Comparing the initial functions q_i, u and the results of calculations, we can estimate the convergence of the algorithm.

The characteristics of the computer are as follows: Processor: Intel(R) Xeon(R) CPU E5-2678 v3 @ 2.50GHz (2 two processors); RAM: 64.0 GB; OC: Windows 10 Pro x64.

For brevity, we only display graphics and tables with the results of calculations q_i. Assume that all coefficients in the equation (4) are known. Every experiment consists of the following steps: definition of the points $\{y_i\}_{i=1}^r$ and the functions q_i, Φ_i $(i = 1, 2, \ldots, r)$; defining the parameters of a grid; converting all used functions into arrays in accordance with grid nodes and the time step; solving the direct problem (4), (5), constructing the functions $\psi_i(t)$ $(i = 1, 2, \ldots, r)$ and adding random noise to the values of these functions; solving the inverse problem (3) – (5); comparing the initial data and the results of calculations of the functions q_i and u.

Consider the results of calculations for the first group of data. Our domain is a cube with the unit edge, whose diametrically opposite corners have the coordinates $(0; 0; 0)$ and $(1; 1; 1)$.

The first group of the data. Let $r = 3$ and let the points y_i have the coordinates: $(0,25; 0,75; 0,25)$, $(0,75; 0,5; 0,75)$, and $(0,5; 0,25; 0,5)$. The functions q_i are the functions $q_1 = 2t+1$, $q_2 = (t-1)^2$, and $q_3 = t^3+3$. The number of the time steps is equal to $M = 10$.

1) Since the problem is three-dimensional, it is necessary to partition the domain into tetrahedra. Let us denote the steps in spatial variables by Δx, Δy, and Δz. Let us construct a grid on part of the boundary $x_3 = 0$ consisting of right triangles with legs equal to Δx and Δy. Next, we duplicate this layer by raising it to Δz and connecting the points, thus obtaining rectangular tetrahedra of the height Δz with a right triangle at the base. We employ three grids Z_0, Z_1, and Z_2 with the number of nodes N equal 729, 2197, and 9261, respectively.

2) Next, we define the arrays of nodes on the faces of the cube. Note that we use the homogeneous Dirichlet condition on all faces except for the lower face of the cube.

3) The time step is equal to $\tau = T/M$. Introduce the coefficients as follows: $a = (t+1)*(x+y+z+1)$, $b_1 = (t+1)*(x+1)^2$, $b_2 = (t+1)*(y+1)^2$, $b_3 = (t+1)*(z+1)^2$, $c_{11} = (x+1)/(t+1)$, $c_{11} = (t+1)*(x+1)^2$, $c_{22} = (y+1)/(t+1)$, $c_{33} = (z+1)/(t+1)$. Next, we define the right-hand side $f = 1$, the functions g and q_i.

4) The functions Φ_i in our calculations are actually a partition of unity on the lower face of the cube, we divide the boundary $x_3 = 0$ into r parts according to the following rule: the nodes of the ith subdomain are closer to the ith point y_i than to all other points. So $\Phi_i = 1$ on some collection of nodes and vanishes at the remaining nodes.

5) Next, we solve a direct problem (4), (5), as it was described in the previous section. The next step is constructing the functions $\psi_i = u(t, y_i)$ $(i = 1, 2, \ldots, r)$. To add a random perturbation, we employ a uniformly distributed random variable $noise \in (-\delta; \delta)$ $(\delta \in [0, 1])$ with zero mean, so 100δ is a deviation in percents. The resulting functions are of the form $\psi_i(j\tau) = \psi_i(j\tau)(1 + noise(j\tau))$. For the first group of the data we take $\delta = 0$.

6) Introduce the calculation errors as follows: the equality $\varepsilon_q = \max_i(\max_j |q^j(i\tau) - q_i^j|)$ defines the calculation error for the functions q_i (the numbers q_i^j are the result of calculations, $q_i^j \approx q^j(\tau i)$, $j = 1, \ldots, r$); the error of calculations of a solution u is defined as $\varepsilon_u = \max_{i,j} |u_{i,j} - u(p_i, \tau j)|$, where $i = 1, 2, \ldots, N$ and $j = 1, 2, \ldots, M$. Let τ_s be execution time of the algorithm, including the time for solving the direct problem, in seconds.

The results of calculations for the three grids (the case of $\delta = 0$) show that the graphics of the initial and the constructed functions actually coincide, so we do not display the results. The quantities ε_α, ε_u, and τ_s for the above three grids are as follows: $(2,4e^{-14},4,9e^{-15},1.67)$, $(4,9e^{-14},1,4e^{-14},11,9)$, $(5,6e^{-14},1,3e^{-14},373,3)$.

For the second group of experiments, we take only one point y_1 and add 1, 5 and 10 percent noise. The number of nodes of the grid is equal to 1331. Changing the coordinates of the point y_1 with the step 0.1 from $(0,1;0,1;0,1)$ to $(0,9;0,9;0,9)$, we obtain practically identical result and the average parameters are as follows: $\varepsilon_q^{avr} = 5,55e^{-16}$, $\varepsilon_u^{avr} = 3,82e^{-16}$, $\tau_s^{avr} = 3,22$. The largest error achieves at the point $y_1 = (0,5;0,5;0,5)$. In the next table we take $y_1 = (0,5;0,5;0.5)$. The Table 1 shows the dependence of the errors on the functions q_i and a random noise. Next, we take $M = 100$

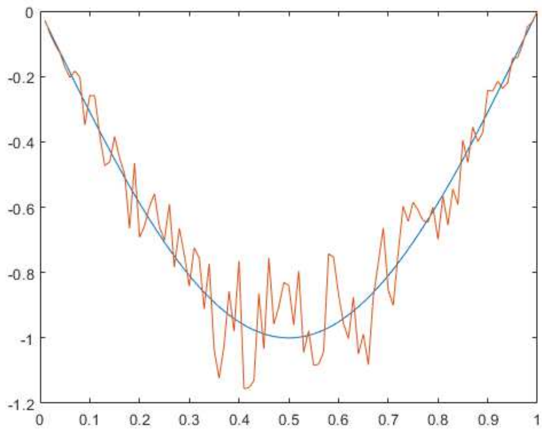

Fig. 1. The results of calculations of q_1 with 25% noise

and $\delta = 0,25$. The results are displayed on Fig. 1 for the function $q_1 = \sin(\pi(t+1))$. In this case $\varepsilon_q = 0,24$, $\varepsilon_u = 0,091$, $\tau_s = 32,8$. The calculation shows that the algorithm is stable with respect to the noise.

Table 1

The results of experiments for the second group of the data

No	δ	q_i	ε_q	ε_u	τ_s
1	0,01	$\log(t+1)$	0,0061	0,0033	3,41
2	0,05	$\log(t+1)$	0,0185	0,007	3,46
3	0,1	$\log(t+1)$	0,0581	0,031	3,44
4	0,01	e^{t+1}	0,044	0,24	3,65
5	0,05	e^{t+1}	0,217	0,09	3,3
6	0,1	e^{t+1}	0,398	0,19	3,33
7	0,01	$\sin(\pi(t+1))$	0,0091	0,0037	3,46
8	0,05	$\sin(\pi(t+1))$	0,037	0,015	3,25
9	0,1	$\sin(\pi(t+1))$	0,089	0,032	3,34

For the third group of experiments, we use an array of 8 points $\{y_i\}$ and the corresponding functions q_i below. We also slightly change the mesh construction area by stretching and compressing it by 2 times relative to the X and Z axes, respectively. Let's set the random noise to 5 percent. The results are exhibited in Table 2 and on Fig. 2. According to the results of computational experiments, it is clear that the calculation error increases as the coordinates of the overdetermination point move away from the lower face. The results shows good convergence of the algorithm as a whole.

Table 2

The results of experiments for the third group of the data

No	y_i	q_i	ε_{q_i}
1	(0,2;0,1;0,45)	$\sin(\pi(10t+1))-16$	2,44
2	(0,6;0,3;0,35)	$(t-2)^2+16$	1,39
3	(1;0,5;0,25)	$(t-1)^3-12$	1,32
4	(1,4;0,7;0,35)	$\log^2(0,1t+1)-8$	4,53
5	(0,2;0,9;0,05)	$2t+12$	0,72
6	(0,6;0,7;0,15)	$-10t-1$	0,31
7	(1,8;0,1;0,05)	$-\cos(\pi 10t)+8$	0,93
8	(1,4;0,3;0,15)	$-e^{2t+0,5}+4$	0,97

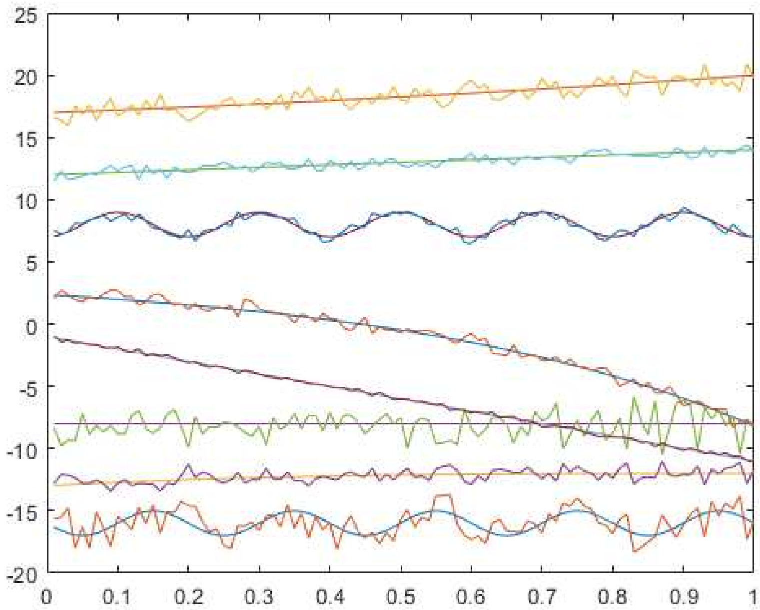

Fig. 2. The results of calculations of q_i with 5% noise

Conclusions

Using theoretical results on well-posedness of the problem, we construct a numerical algorithm for recovering the surface flow on the lower face with the use of point observations of the concentration. It is based on the conventional methods (in our case FEM and difference schemes). The results of numerical experiments are presented. The obtained results reveal the accuracy, efficiency, and robustness of the proposed algorithm. It is stable under random perturbations of the data.

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ВОССТАНОВЛЕНИЕ ПОТОКА НА ГРАНИЦЕ ЭКОСИСТЕМА-АТМОСФЕРА

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Мы рассматриваем модели тепломассопереноса в квазистационарном случае, т.е. все коэффициенты и данные зависят от времени, но производная по времени в уравнении отсутствует. Исследуется обратная задача восстановления потока на границе области по заданным значениям решения в наборе точек, лежащим внутри области. Поток ищется в виде конечного отрезка ряда Фурье, с неизвестными коэффициентами. Задача определения коэффициентов сводится с помощью специальных решений сопряженной задачи к системе алгебраических уравнений. Исходное уравнение рассматривается в цилиндрической пространственной области. Это выбор сделан в силу того, что этот случай, как правило рассматривается в приложениях. Доказана теоремы существования и единственности решений прямой задачи. Полученные результаты используются в доказательстве соответствующих результатов для обратной задачи. В трехмерном случае строится численный алгоритм и приводятся результаты численных экспериментов. Показывается, что алгоритм устойчив к случайным возмущениям данных. Используется метод конечных элементов. Результаты могут быть использованы, например, в задачах определения потоков парниковых газов из почвы по данным замерам концентраций.

Ключевые слова: обратная задача; параболическое уравнение; поток; тепломассоперенос.

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