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DIFFERENTIAL EQUATIONS OF ELLIPTIC TYPE WITH VARIABLE OPERATORS AND GENERAL ROBIN BOUNDARY CONDITION IN UMD SPACES

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In this paper we study an abstract second order differential equation of elliptic type with variable operator coefficients and general Robin boundary conditions containing an unbounded linear operator. The study is performed when the second member belongs to a Sobolev space and uses the celebrated Dore–Venni theorem. Here, we do not assume the differentiability of the resolvent operators. We give necessary and sufficient conditions on the data to obtain existence, uniqueness of the classical solution satisfying the maximal regularity property are obtained under the Labbas–Terreni assumption. Our techniques use essentially the semigroups theory, fractional powers of linear operators, Dunford's functional calculus and interpolation theory. This work is naturally the continuation of the ones studied by R. Haoua in the UMD spaces and homogenous cases. We also give an example to which our theory applies.

Keywords: second-order abstract elliptic differential equations; Robin boundary conditions; analytic semigroup; maximal regularity; Dunford operational calculus.

1. Introduction and Hypotheses

In a complex Banach space E, we consider the second-order differential equation with variable operator coefficients

$$u''(x) + A(x)\,u(x) - \omega u(x) = f(x), \qquad x \in (0,1)\,, \tag{1}$$

under the Dirichlet boundary conditions

$$u(1) = u_1, \tag{2}$$

and the abstract Robin boundary conditions

$$u'(0) - Hu(0) = d_0. \tag{3}$$

Here ω is a positive real number, d_0, u_1 are given elements of E, $(A(x))_{x\in[0,1]}$ is a family of closed linear operators whose domains $D(A(x))$ are dense in E, H is a closed linear operator in E, and f belongs to $L^p(0,1;E)$ where $1 < p < +\infty$. This article extends and improves the studies done in [1], where the authors have studied (1) – (3) under the general Robin homogeneous boundary value conditions, in the framework of UMD spaces, where we study the existence, the uniqueness, and the maximal regularity of the classical solution of problem (1) – (3). In particular, we give necessary and sufficient conditions to obtain a unique classical solution of problem (1) – (3) satisfying maximal regularity. We consider some fixed ω_0 and we set, for $\omega \geq \omega_0$, $x \in [0,1]$,

$$A_\omega(x) = A(x) - \omega I.$$

Our aim is to find a classical solution u to (1) – (3), i.e, a function u such that

$$\begin{cases} \text{for a.e } x \in (0,1), \quad u(x) \in D(A(x)) \text{ and} \\ x \mapsto A(x)\, u(x) \in L^p(0,1;E) \\ u \in W^{2,p}(0,1;E), \end{cases} \tag{4}$$

u satisfies $u(0) \in D(H)$ and (1) – (3). Generally, more conditions are needed on f or on E. Here we will assume that

$$E \text{ is a UMD space.} \tag{5}$$

We recall that a Banach space E is UMD if and only if for some $p > 1$ (and thus for all p) the Hilbert transform is continuous from $L^p(\mathbb{R};E)$ into itself, see Bourgain [2], Burkholder [3]. Throughout this work we suppose that the family of closed linear operators $(A(x))_{x\in[0,1]}$ satisfies

$\exists\omega_0 > 0$, $\exists C > 0 : \forall x \in [0,1]$, $\forall z \geq 0$, $(A_{\omega_0}(x) - zI)^{-1} \in \mathcal{L}(E)$ and

$$\left\|(A_{\omega_0}(x) - zI)^{-1}\right\|_{\mathcal{L}(E)} \leq \frac{C}{1+z}. \tag{6}$$

This assumption means exactly the ellipticity of our problem in the sense of Krein [4]. It follows that for $x \in [0,1]$, $\omega \geq \omega_0$ the square roots

$$Q_\omega(x) = -(-A_\omega(x))^{1/2},$$

are well defined and generate analytic semigroups $\left(e^{yQ_\omega(x)}\right)_{y>0}$ not necessarily strongly continuous in 0 see Balakrishnan [5] for dense domains and Martinez-Sanz [6] for non dense domains. We will assume, moreover:

$\exists C \geq 1$, $\theta_0 \in]0,\pi[$: $\forall s \in \mathbb{R}$, $\forall x \in [0,1]$, $\forall \omega \geq \omega_0$, $(-A_\omega(x))^{is} \in \mathcal{L}(E)$ and

$$\left\|(-A_\omega(x))^{is}\right\|_{\mathcal{L}(E)} \leq Ce^{\theta_0|s|}, \tag{7}$$

$\exists C$, α, $\mu > 0$: $\forall x$, $\tau \in [0,1]$, $\forall z \geq 0$, $\forall \omega \geq \omega_0$,

$$\begin{cases} \left\|A_\omega(x)(A_\omega(x) - zI)^{-1}\left(A_\omega(x)^{-1} - A_\omega(\tau)^{-1}\right)\right\|_{\mathcal{L}(E)} \leq \dfrac{C|x-\tau|^\alpha}{|z+\omega|^\mu} \\ \text{with } \alpha + 2\mu - 2 > 0. \end{cases} \tag{8}$$

This hypothesis is known as the Labbas–Terreni assumption. Operators H and $Q_\omega(x)$ have to satisfy

$\exists C > 0$: $\forall x \in [0,1]$, $\forall \omega \geq \omega_0$, $\forall \xi \in E$,

$$\left\|\left[(Q_\omega(x) - H)^{-1} - (Q_\omega(0) - H)^{-1}\right]\xi\right\|_{\mathcal{L}(E)} \leq Cx^{\alpha+2\mu}\|\xi\|_E, \tag{9}$$

with $\alpha + 2\mu > 2$ and the following commutativity conditions

$\forall x \in [0,1]$, $\forall \omega \geq \omega_0$,

$$(Q_\omega(x))^{-1}(Q_\omega(x) - H)^{-1} = (Q_\omega(x) - H)^{-1}(Q_\omega(x))^{-1}, \tag{10}$$

and

$$d_0 \in D(Q(0)). \tag{11}$$

Remark 1. From (6) we deduce that, there exists $\theta_0 \in \left]0, \frac{\pi}{2}\right[$ and $r_0 > 0$ such that for all x belonging to $[0,1]$, the resolvent of $(A_{\omega_0}(x))$ verifies:

$$\rho(A_{\omega_0}(x)) \supset \Pi_{\theta_0, r_0} = \{z \in \mathbb{C} \setminus \{0\} : |\arg z| \leq \theta_0\} \cup \overline{B(0, r_0)},$$

where $\overline{B(0, r_0)}$ is the closed ball of radius r_0 and centered in 0. We denote by Γ the boundary of Π_{θ_0, r_0} oriented from $\infty e^{i\theta_0}$ to $\infty e^{-i\theta_0}$.

Equation (1) has been studied by several authors via various approaches. In the constant case of operators $A(x) = A$, many authors dealt with partial differential equations with non-local boundary conditions. We can first refer to the pioneering works by T. Carleman [7] who in the thirties used singular integral technique to handle an elliptic equation in which boundary values of unknown function on two different points are related. This was the starting point of many studies, for example, [8–11]. The next step was the important paper of Bitsadze and Samarskii [12], in 1969, where the authors analyzed an elliptic equation with unknown functions on the boundary connecting its values at some points on the boundary with other points in the interior of the domain. This problem models some phenomena occurring in plasma physics. Paper [13] gave rise to many works on non-local boundary value problems using different techniques. Let us mention a systematic study done by Skubachevskii [12] and Gurevich [14] and references therein. Yakubov [15] and some others [16, 17] use the operator-differential equation tools to study some classes of elliptic partial differential equations with non-local boundary conditions. The Robin condition was treated by M. Cheggag *et al* [18] in a commutative framework, when $f \in L^p(0,1;E)$ with $1 < p < +\infty$. They considered that the spectral parameter which appears in the boundary conditions is zero, and gave interesting results for this problem when E is an UMD space where they proved that the problem has a unique classical solution $u \in W^{2,p}(0,1;E) \cap L^p(0,1;D(A))$ such as $u(0) \in H$ if and only if d_0, u_1 are in the interpolation space $(D(A), E)_{\frac{1}{2p}+\frac{1}{2},p}$, $(D(A), E)_{\frac{1}{2p},p}$ respectively. The same authors, in [19], studied the problem (1) – (3), but this time, in the absence of the spectral parameter $\omega = 0$, in the same commutative frame, in the same commutative setting and in a Hölder space. In other words, they assumed that f belongs to $C^\theta([0,1];E)$ with $\theta \in]0,1[$, and under certain assumptions about the operator A, they studied existence, uniqueness and maximal regularity and then gave some positive results for this problem. They show that the problem (1) – (3) has a unique strict solution $u \in C^2([0,1];E) \cap C([0,1];D(A))$ such as $u(0) \in H$, satisfying the maximal regularity property u'', $Au \in C^\theta([0,1];E)$, if and only if $u_1 \in D(A)$, $d_0 \in D(Q)$ and Qd_0, $f(0), -Au_1 + f(1)$ are in the interpolation space $(D(Q), X)_{1-\theta,\infty}$, with $Q = -\sqrt{-A}$.

In the variable case of operators $A(x)$, the commutator hypothesis (8) was used for the first time in Labbas [20] for the same problem but with boundary conditions of Dirichlet type, in Bouziani *et al* [21] for transmission conditions and Haoua *et al* [22] and [1] for Robin conditions. All these studies were performed in the frame work of höelderian spaces. For the bounded interval, a direct method based on Dunford's operational calculus has been used in Labbas [20] under some hypotheses on differentiability of resolvent of operators $A_\omega(x)$. Moreover, the case of differentiability of the resolvent of $(A(x))_{x\in[0,1]}$ was used by Da Prato and Grisvard [23], Labbas [20] and Boutaous *et al* [24] . Also, in these studies, the boundary conditions considered were of Dirichlet type. However, in Boutaous *et al* [24] the authors used the Krein's approach, under some natural differentiability assumptions on the resolvent of the square roots $Q_\omega(x)$ combining those of Yagi [25] and Acquistapace–Terreni [26].

The organization of the paper is as follows. Section 2, contains some technical lemmas which will be useful for the study of problem (1) – (3). In Section 3, an heuristic reasoning is used to obtain a representation of the solution. We obtain an integral equation which is solved using (8). Section 4 is devoted to the study of the maximal regularity of the solution; we give necessary and sufficient compatibility conditions to obtain it. In section 5, the existence of the solution is proved using the associated approximating problem. Finally, in section 6, we provide an example to which our abstract results apply.

2. Technical Lemmas

Lemma 1. *There exists $C > 0$ such that for each $z \in \Gamma$ and $r > 0$, we have*

$$|z + r| \geq C\,|z|\,, |z + r| \geq Cr, |z - r| \geq C\,|z|\,, |z - r| \geq Cr,$$

and

$$\forall r > 0,\ \forall \nu \in [0,1]\,,\ \int_\Gamma \frac{|dz|}{|z \pm r|\,|z|^\nu} \leq \frac{C}{r^\nu}.$$

Proof. See [27, Lemma 6.1 and 6.2, p. 564].

□

Lemma 2. *Assume that (6) hold. There exists a constant $M \geq 0$ and $\omega_1^* > \omega_0$ such that, for all $\omega \geq \omega_1^*$ and $x \in [0,1]$, operators $I \pm e^{2Q_\omega(x)}$ are invertible in $\mathcal{L}(E)$ and*

$$\left\|\left(I \pm e^{2Q_\omega(x)}\right)^{-1}\right\|_{\mathcal{L}(E)} \leq M.$$

Proof. Let $x \in [0,1]$. Since $Q_\omega(x)$ generates a bounded analytic semigroup and $0 \in \rho(Q_\omega(x))$, there exist $M \geq 1$ and $\delta > 0$ such that for any $y > 0$ and $\omega > 0$, we have

$$\left\|e^{yQ_\omega(x)}\right\|_{\mathcal{L}(E)} \leq Me^{-y\delta},$$

see [28, Theorem 6.13, p. 74] and in the case of non dense domains see [29, Proposition 2.1.1, p. 35 and Proposition 2.3.1, p. 55,56]. We can choose $k \in \mathbb{N}^*$ such that

$$K_1 e^{-2kn_1\delta} \leq \frac{1}{2} < 1.$$

Then $I - e^{2kQ_\omega(x)}$ is boundedly invertible with

$$\left\|\left(I - e^{2kQ_\omega(x)}\right)^{-1}\right\|_{\mathcal{L}(E)} \leq \frac{1}{1 - 1/2} = 2,$$

so $0 \in \rho\left(I - e^{2Q_\omega(x)}\right)$ since

$$\begin{aligned} I &= \left(I - e^{2Q_\omega(x)}\right)\left(I + e^{2Q_\omega(x)} + ... + e^{2(k-1)Q_\omega(x)}\right)\left(I - e^{2kQ_\omega(x)}\right)^{-1} = \\ &= \left(I - e^{2kQ_\omega(x)}\right)^{-1}\left(I + e^{2Q_\omega(x)} + ... + e^{2(k-1)Q_\omega(x)}\right)\left(I - e^{2Q_\omega(x)}\right). \end{aligned}$$

Moreover,

$$\left\| \left(I - e^{2kQ_\omega(x)}\right)^{-1} \right\|_{\mathcal{L}(E)} \leq$$
$$\leq \left(I + \left\|e^{2Q_\omega(x)}\right\|_{\mathcal{L}(E)} + ... + \left\|e^{2Q_\omega(x)}\right\|^{k-1}_{\mathcal{L}(E)}\right) \left\| \left(I - e^{2kQ_\omega(x)}\right)^{-1} \right\|_{\mathcal{L}(E)} \leq 2K_1^k.$$

We obtain the result for $I + e^{2Q_\omega(x)} = I - \left(-e^{2Q_\omega(x)}\right)$ if we replace $e^{2kQ_\omega(x)}$, $e^{2Q_\omega(x)}$ by $-e^{2kQ_\omega(x)}$, $-e^{2Q_\omega(x)}$ in the above proof.

□

Lemma 3. *Assume that (6) hold. Then there exist constants $C > 0$, $\omega_1^* > \omega_0$ such that for all $\omega \geq \omega_1^*$ and $x \in [0, 1]$, we have*

$$\left\|(Q_\omega(x) - zI)^{-1}\right\|_{\mathcal{L}(E)} \leq \frac{C}{\sqrt{1 + |\omega| + |z|}}.$$

Proof. Using [4, p. 116, 117] and for all $z \geq 0$ and $x \in [0, 1]$, we have

$$(Q_\omega(x) - zI)^{-1} = -\left(\sqrt{-A_\omega(x)} + zI\right)^{-1} =$$
$$= \frac{-1}{2\pi i} \int_\Gamma \frac{(-A_\omega(x) - \lambda I)^{-1}}{z + \sqrt{\lambda}} d\lambda = \frac{-1}{\pi} \int_0^{+\infty} \frac{\sqrt{s}\,(-A(x) + \omega I + sI)^{-1}}{s + z^2} ds.$$

Due to hypothesis (6) and Lemma 1, we obtain

$$\left\|(Q_\omega(x) - zI)^{-1}\right\|_{\mathcal{L}(E)} \leq C \int_0^{+\infty} \frac{\sqrt{s}}{(1 + |\omega| + s)(s + z^2)} ds \leq$$
$$\leq C \int_0^{+\infty} \frac{t^2}{(1 + |\omega| + t^2)(t^2 + z^2)} dt \leq$$
$$\leq \frac{C}{1 + |\omega| - z^2} \left[\int_0^{+\infty} \frac{1 + |\omega|}{1 + |\omega| + t^2} dt - \int_0^{+\infty} \frac{z^2}{t^2 + z^2} dt\right] \leq \frac{C}{\sqrt{1 + |\omega| + |z|}}.$$

□

Lemma 4. *Assume that (5) – (10) hold. Then there exist constants $C > 0$, $\omega_1^* > \omega_0$ such that for all $\omega \geq \omega_1^*$, and $x \in [0, 1]$, operator $Q_\omega(x) \pm H$ is boundedly invertible and*

$$\left\|(Q_\omega(x) \pm H)^{-1}\right\|_{\mathcal{L}(E)} \leq \frac{C}{\sqrt{\omega}}.$$

Proof. See [19, Proposition 7, p. 987].

□

For $\omega \geq \omega_1^*$, we also define the linear operator $\Lambda_\omega(x)$ by

$$\begin{cases} D(\Lambda_\omega(x)) = D(Q_\omega(x)) \\ \Lambda_\omega(x) = Q_\omega(x) - H + e^{2Q_\omega(x)}(Q_\omega(x) + H),\ x \in [0, 1], \end{cases}$$

which will be the determinant of the system of our problem.

Lemma 5. *Assume (5) – (7) and (10). Then for all $\omega \geq \omega_1^*$ and $x \in [0, 1]$, $\Lambda_\omega(x)$ is closed and boundedly invertible with*

$$[\Lambda_\omega(x)]^{-1} = (Q_\omega(x) - H)^{-1} [I + M_\omega(x)]^{-1} \left(I - e^{2Q_\omega(x)}\right)^{-1},$$

where

$$M_\omega(x) = 2\left(I - e^{2Q_\omega(x)}\right)^{-1} Q_\omega(x)\, e^{2Q_\omega(x)} \left(Q_\omega(x) - H\right)^{-1},$$

and

$$\left[\Lambda_\omega(x)\right]^{-1} = \left(Q_\omega(x) - H\right)^{-1} + \left(Q_\omega(x) - H\right)^{-1} W(x),$$

with

$$W(x) \in \mathcal{L}(E), \quad \left(Q_\omega(x) - H\right)^{-1} W(x) = W(x)\left(Q_\omega(x) - H\right)^{-1},$$

and

$$W(x)(E) \subset \bigcap_{k=1}^{+\infty} D\left(Q_\omega(x)^k\right).$$

Proof. See [22, Lemma 2.5, p. 4].

□

Lemma 6. *From (6) and (8), we have*

$$\begin{cases} \exists C, \alpha, \mu > 0 : \forall x, \tau \in [0,1], \forall z \geq 0, \forall \omega \geq \omega_1^* \\ \left\| Q_\omega(x)\left(Q_\omega(x) - zI\right)^{-1}\left(Q_\omega(x)^{-1} - Q_\omega(\tau)^{-1}\right)\right\|_{L(E)} \leq \dfrac{C\left|x-\tau\right|^\alpha}{\left|z+\omega\right|^\mu} \\ \text{with } \alpha + 2\mu - 2 > 0. \end{cases}$$

Proof. See [1, Lemma 4, p. 22].

□

3. Representation of the Solution

Assume that there exists a solution u of (1) – (3) satisfying (4). Setting when $A_\omega(x) = A - \omega I$ is a constant operator satisfying the natural ellipticity hypothesis is mentioned above (we will take $Q_\omega = -\left(-A_\omega\right)^{1/2}$). By using the method based on the variation of constant and Green's functions, the solution of problem (1) – (3) is (see [18])

$$\begin{gathered} u(x) = e^{xQ_\omega}\left[\Lambda_\omega^{-1} d_0 + \left(Q_\omega + H\right)\Lambda_\omega^{-1} e^{Q_\omega} u_1\right] + \frac{1}{2} e^{xQ_\omega}\left(Q_\omega + H\right)\Lambda_\omega^{-1} Q_\omega^{-1}\int_0^1 e^{sQ_\omega} f(s)\, ds - \\ -\frac{1}{2} e^{xQ_\omega}\left(Q_\omega + H\right)\Lambda_\omega^{-1} e^{Q_\omega} Q_\omega^{-1}\int_0^1 e^{(1-s)Q_\omega} f(s)\, ds + e^{(1-x)Q_\omega}\left[\left(I - \left(Q_\omega + H\right)\Lambda_\omega^{-1} e^{2Q_\omega}\right) u_1 - \right. \\ \left. -\Lambda_\omega^{-1} e^{Q_\omega} d_0\right] - \frac{1}{2} e^{(1-x)Q_\omega}\left(Q_\omega + H\right)\Lambda_\omega^{-1} e^{Q_\omega} Q_\omega^{-1}\int_0^1 e^{sQ_\omega} f(s) - \\ -\frac{1}{2} e^{(1-x)Q_\omega}\left[I - \left(Q_\omega + H\right)\Lambda_\omega^{-1} e^{2Q_\omega}\right] Q_\omega^{-1}\int_0^1 e^{(1-s)Q_\omega} f(s)\, ds + \\ +\frac{1}{2} Q_\omega^{-1}\int_0^x e^{(x-s)Q_\omega} f(s)\, ds + \frac{1}{2} Q_\omega^{-1}\int_x^1 e^{(s-x)Q_\omega} f(s)\, ds. \end{gathered}$$

Set

$$L_{Q_\omega(x)}(x,f)=\frac{1}{2}e^{xQ_\omega(x)}(Q_\omega(x)+H)(\Lambda_\omega(x))^{-1}Q_\omega(x)^{-1}\int_0^1 e^{sQ_\omega(x)}f(s)\,ds-$$
$$-\frac{1}{2}e^{xQ_\omega(x)}(Q_\omega(x)+H)(\Lambda_\omega(x))^{-1}e^{Q_\omega(x)}Q_\omega^{-1}(x)\int_0^1 e^{(1-s)Q_\omega(x)}f(s)-$$
$$-\frac{1}{2}e^{(1-x)Q_\omega(x)}(Q_\omega(x)+H)(\Lambda_\omega(x))^{-1}e^{Q_\omega(x)}Q_\omega(x)^{-1}\int_0^1 e^{sQ_\omega(x)}f(s)\,ds-$$
$$-\frac{1}{2}e^{(1-x)Q_\omega(x)}\left[I-(Q_\omega(x)+H)(\Lambda_\omega(x))^{-1}e^{2Q_\omega(x)}\right]Q_\omega^{-1}(x)\int_0^1 e^{(1-s)Q_\omega(x)}f(s)+$$
$$+\frac{1}{2}Q_\omega(x)^{-1}\int_0^x e^{(x-s)Q_\omega(x)}f(s)\,ds+\frac{1}{2}Q_\omega(x)^{-1}\int_x^1 e^{(s-x)Q_\omega(x)}f(s)\,ds.$$

We can write that:

$$L_{Q_\omega(x)}(x,f)=L_{Q_\omega(x)}(x,u''(x)+A_\omega(x)u(x)),$$

After two integrations by parts and some formal calculus, as in R. Haoua and A. Medeghri [22], we obtain the following abstract equation:

$$w+P_\omega w=G(x,f),$$

where

$$w(\cdot)=A_\omega(\cdot)u(\cdot).$$

Here, for all $x\in[0,1]$, $\omega\geq\omega_1^*$

$$(P_\omega w)(x)=\frac{1}{2}K_\omega(x)e^{xQ_\omega(x)}\int_0^1 Q_\omega(x)^3e^{sQ_\omega(x)}\left(Q_\omega(s)^{-2}-Q_\omega(x)^{-2}\right)w(s)\,ds-$$
$$-\frac{1}{2}e^{xQ_\omega(x)}K_\omega(x)e^{Q_\omega(x)}\int_0^1 Q_\omega(x)^3e^{(1-s)Q_\omega(x)}\left(Q_\omega(s)^{-2}-Q_\omega(x)^{-2}\right)w(s)\,ds-$$
$$-\frac{1}{2}e^{(1-x)Q_\omega(x)}K_\omega(x)e^{Q_\omega(x)}\int_0^1 Q_\omega(x)^3e^{sQ_\omega(x)}\left(Q_\omega(s)^{-2}-Q_\omega(x)^{-2}\right)w(s)\,ds+$$
$$+\frac{1}{2}e^{(1-x)Q_\omega(x)}K_\omega(x)e^{2Q_\omega(x)}\int_0^1 Q_\omega(x)^3e^{(1-s)Q_\omega(x)}\left(Q_\omega(s)^{-2}-Q_\omega(x)^{-2}\right)w(s)\,ds-$$
$$-\frac{1}{2}e^{(1-x)Q_\omega(x)}\int_0^1 Q_\omega(x)^3e^{(1-s)Q_\omega(x)}\left(Q_\omega(s)^{-2}-Q_\omega(x)^{-2}\right)w(s)\,ds+$$
$$+\frac{1}{2}\int_0^x Q_\omega(x)^3e^{(x-s)Q_\omega(x)}\left(Q_\omega(s)^{-2}-Q_\omega(x)^{-2}\right)w(s)\,ds$$
$$+\frac{1}{2}\int_x^1 Q_\omega(x)^3e^{(s-x)Q_\omega(x)}\left(Q_\omega(s)^{-2}-Q_\omega(x)^{-2}\right)w(s)\,ds=\sum_{i=1}^{7}I_i(x),$$

where

$$K_\omega(x)=(Q_\omega(x)+H)[\Lambda_\omega(x)]^{-1},$$

and

$$G_{Q_\omega(x)}(d_0,u_1,f)(x)=-A_\omega(x)L_{Q_\omega(x)}(x,f)-A_\omega(x)e^{xQ_\omega(x)}\left[(\Lambda_\omega(x))^{-1}d_0+\right.$$
$$\left.+K_\omega(x)e^{Q_\omega(x)}u_1\right]-A_\omega(x)e^{(1-x)Q_\omega(x)}\left[\left(I-K_\omega(x)e^{2Q_\omega(x)}\right)u_1-(\Lambda_\omega(x))^{-1}e^{Q_\omega(x)}d_0\right].$$

Proposition 1. *Assume (5) – (10). Then there exists $\omega_1^* > 0$ such that for all $\omega \geq \omega_1^*$:*

$$\|P_\omega\|_{\mathcal{L}(L^p(0,1;E))} \leq \frac{1}{2}.$$

Proof. See [1, Proposition 2, p. 25].

□

Therefore for all $\omega \geq \omega_1^*$, $\|P_\omega\|_{\mathcal{L}(L^p(0,1;E))} \leq \dfrac{1}{2}$ which leads us to invert $I + P_\omega$ in the space $L^p\,(0,1;E)$.

We can write for all $\omega \geq \omega_1^*$ and $x \in [0,1]$

$$u\,(x) = A_\omega\,(x)^{-1}\,(I + P_\omega)^{-1}\,G_{Q_\omega(x)}\,(d_0, u_1, f)\,(x)\,. \tag{12}$$

4. Regularity of the Solution

Throughout this section we assume that $\omega \geq \omega_1^*$.

4.1. Regularity of the Second Member $G_{Q_\omega(x)}\,(d_0, u_1, f)$

For convenience we present the results below in the form of lemmas.

Lemma 7. *Assume (5) – (7) and $f \in L^p\,(0,1;E)$ with $1 < p < \infty$. Then for all $\omega \geq \omega_1^*$*

1) $t \longmapsto Q_\omega\,(x) \int_0^t e^{(t-s)Q_\omega(x)} f\,(s)\,ds \in L^p\,(0,1;E)$;

2) $t \longmapsto Q_\omega\,(x) \int_t^1 e^{(s-t)Q_\omega(x)} f\,(s)\,ds \in L^p\,(0,1;E)$;

3) $t \longmapsto Q_\omega\,(x) \int_0^1 e^{(t+s)Q_\omega(x)} f\,(s)\,ds \in L^p\,(0,1;E)$;

4) $\int_0^1 e^{sQ_\omega(x)} f\,(s)\,ds \in (D\,(Q\,(x))\,,E)_{\frac{1}{p},p}$ *and* $\int_0^1 e^{(1-s)Q_\omega(x)} f\,(s)\,ds \in (D\,(Q\,(x))\,,E)_{\frac{1}{p},p}$.

Proof. See [30].

□

We have the following lemmas as in [31]

Lemma 8. *Fix $x \in [0,1]$, $p \in\,]1,\infty[$ and $\omega \geq \omega_1^*$. Then*

1) $t \longmapsto A_\omega\,(x)\,e^{tQ_\omega(x)}\varphi \in L^p\,(0,1;E)$ *if and only if* $\varphi \in (D\,(A\,(x))\,,E)_{\frac{1}{2p},p}$;

2) $t \longmapsto Q_\omega\,(x)\,e^{tQ_\omega(x)}\varphi \in L^p\,(0,1;E)$ *if and only if* $\varphi \in (D\,(A\,(x))\,,E)_{\frac{1}{2p}+\frac{1}{2},p}$.

In the following, it is important to note that

$$(D\,(A\,(x))\,,E)_{\frac{1}{2p},p} \subset D\,(Q\,(x)) \subset (D\,(A\,(x))\,,E)_{\frac{1}{2p}+\frac{1}{2},p}\,.$$

This is due to the reiteration property

i) $(D(A(x)),E)_{\frac{1}{2p},p} = (E,D(A(x)))_{1-\frac{1}{2p},p} = \left(E,D\left(Q(x)^2\right)\right)_{1-\frac{1}{2p},p} =$
$$= (E,D(Q(x)))_{2-\frac{1}{2p},p} = \left\{\psi \in E : Q\psi \in (E,D(Q(x)))_{1-\frac{1}{2p},p}\right\}.$$

ii) $(D(A(x)),E)_{\frac{1}{2p}+\frac{1}{2},p} = (E,D(A(x)))_{\frac{1}{2}-\frac{1}{2p},p} = \left(E,D\left(Q(x)^2\right)\right)_{\frac{1}{2}-\frac{1}{2p},p} =$
$$= (E,D(Q(x)))_{1-\frac{1}{p},p} = (D(Q(x)),E)_{\frac{1}{p},p}.$$

Then we obtain the following regularity results of $G_{Q_\omega(x)}(d_0,u_1,f)$.

Proposition 2. *Assume (5) – (11) and $f \in L^p(0,1;E)$ with $1 < p < \infty$. Then for all $\omega \geq \omega_1^*$, $x \longmapsto G_{Q_\omega(x)}(d_0,u_1,f)(x) \in L^p(0,1;E)$ if and only if*

$$\begin{cases} (Q_\omega(0) - H)^{-1} d_0 \in (D(A(0)),E)_{\frac{1}{2p},p} \\ u_1 \in (D(A(1)),E)_{\frac{1}{2p},p}. \end{cases}$$

Proof. Let $x \in [0,1]$ and $\omega \geq \omega_1^*$. We have

$$G_{Q_\omega(x)}(d_0,u_1,f)(x) = A_\omega(x)e^{xQ_\omega(x)}(\Lambda_\omega(x))^{-1}d_0 + A_\omega(x)e^{(1-x)Q_\omega(x)}u_1 -$$
$$-\frac{1}{2}Q_\omega(x)e^{xQ_\omega(x)}K_\omega(x)\int_0^1 e^{sQ_\omega(x)}f(s)\,ds + \frac{1}{2}Q_\omega(x)e^{(1-x)Q_\omega(x)}\int_0^1 e^{(1-s)Q_\omega(x)}f(s)\,ds -$$
$$-\frac{1}{2}Q_\omega(x)\int_0^x e^{(x-s)Q_\omega(x)}f(s)\,ds - \frac{1}{2}Q_\omega(x)\int_x^1 e^{(s-x)Q_\omega(x)}f(s)\,ds + R(x,d_0,u_1,f) =$$
$$= A_\omega(x)e^{xQ_\omega(x)}(\Lambda_\omega(x))^{-1}d_0 + A_\omega(x)e^{(1-x)Q_\omega(x)}u_1 + \sum_{i=1}^4 J_i(x) + R(x,d_0,u_1,f),$$

where

$$R(x,d_0,u_1,f) = A_\omega(x)e^{xQ_\omega(x)}K_\omega(x)e^{Q_\omega(x)}u_1 - A_\omega(x)e^{(1-x)Q_\omega(x)}K_\omega(x)e^{2Q_\omega(x)}u_1 -$$
$$-A_\omega(x)e^{(1-x)Q_\omega(x)}(\Lambda_\omega(x))^{-1}e^{Q_\omega(x)}d_0 + \frac{1}{2}Q_\omega(x)e^{xQ_\omega(x)}K_\omega(x)\int_0^1 e^{(2-s)Q_\omega(x)}f(s)\,ds +$$
$$+\frac{1}{2}Q_\omega(x)e^{(1-x)Q_\omega(x)}K_\omega(x)\int_0^1 e^{(1+s)Q_\omega(x)}f(s)\,ds -$$
$$-\frac{1}{2}Q_\omega(x)e^{(1-x)Q_\omega(x)}K_\omega(x)\int_0^1 e^{(3-s)Q_\omega(x)}f(s)\,ds.$$

For any $\xi \in E$, $k \in \mathbb{N}$, we have $e^{Q_\omega(x)}\xi \in D\left(\left(Q(x)^k\right)\right)$, so

$$A_\omega(x)e^{\cdot Q_\omega(x)}e^{Q_\omega(x)}\xi = e^{\cdot Q_\omega(x)}A_\omega(x)e^{Q_\omega(x)}\xi,$$

and $s \longmapsto A_\omega(x)e^{sQ_\omega(x)}e^{Q_\omega(x)}\xi$ is bounded and thus in $L^p(0,1;E)$. To conclude it is enough to remark that $A_\omega(.)R(.,d_0,u_1,f)$ can be written as a sum of terms $PA_\omega(x)e^{\cdot Q_\omega(x)}e^{Q_\omega(x)}A_\omega(x)$, $PA_\omega(x)e^{(1-\cdot)Q_\omega(x)}e^{Q_\omega(x)}A_\omega(x)$, where $P \in \mathcal{L}(E)$, $\xi \in E$.

For J_3, we consider the following problem:

$$\begin{cases} \psi'(x) - Q_\omega(x)\psi(x) = f(x), \quad .x \in (0,1), \\ \psi(0) = 0. \end{cases} \tag{13}$$

Let ψ be the strict solution of problem (13). Fix $x \in [0,1]$, and set

$$v(s) = e^{(x-s)Q_\omega(x)}\psi(s), \quad s \in [0,x].$$

Then for each $s \in [0,x]$, we have

$$\begin{gathered} v'(s) = -Q_\omega(x)\, e^{(x-s)Q_\omega(x)}\psi(s) + e^{(x-s)Q_\omega(x)}\left[Q_\omega(s)\psi(s) + f(s)\right] = \\ = Q_\omega(x)\, e^{(x-s)Q_\omega(x)}\left[Q_\omega(x)^{-1} - Q_\omega(s)^{-1}\right]Q_\omega(s)\psi(s) + e^{(x-s)Q_\omega(x)}f(s). \end{gathered}$$

Integrating over $]0,x[$ and applying $Q_\omega(x)$ to both sides, we get:

$$\begin{gathered} Q_\omega(x)\psi(x) = \\ = \int_0^x Q_\omega(x)^2 e^{(x-s)Q_\omega(x)}\left[Q_\omega(x)^{-1} - Q_\omega(s)^{-1}\right]Q_\omega(s)\psi(s)\,ds + Q_\omega(x)\int_0^x e^{(x-s)Q_\omega(x)}f(s)\,ds = \\ = \int_0^x Q_\omega(x)^2 e^{(x-s)Q_\omega(x)}\left[Q_\omega(x)^{-1} - Q_\omega(s)^{-1}\right]Q_\omega(s)\psi(s)\,ds + J_3(x); \end{gathered}$$

see [26, p. 56, 57]. Due to [32, Theorem 5.11, p. 59], we have $x \longmapsto Q_\omega(x)\psi(x)$ in $L^p(0,1;E)$ and due to Lemma 7 and Lemma 8, we have

$$x \longmapsto \int_0^x Q_\omega(x)^2 e^{(x-s)Q_\omega(x)}\left[Q_\omega(x)^{-1} - Q_\omega(s)^{-1}\right]Q_\omega(s)\psi(s)\,ds,$$

in $L^p(0,1;E)$. Then $x \longmapsto J_3(x)$ is in $L^p(0,1;E)$.

The same technique is used for the other terms. Therefore, due to Lemma 5, we can write:

$$\begin{gathered} A_\omega(x)\, e^{xQ_\omega(x)}\left(\Lambda_\omega(x)\right)^{-1}d_0 + A_\omega(x)\, e^{(1-x)Q_\omega(x)}u_1 = A_\omega(x)\, e^{xQ_\omega(x)}\left(Q_\omega(x) - H\right)^{-1}d_0 + \\ + A_\omega(x)\, e^{xQ_\omega(x)}\left(Q_\omega(x) - H\right)^{-1}W(x)\,d_0 + A_\omega(x)\, e^{(1-x)Q_\omega(x)}u_1, \end{gathered}$$

where $W(x) \in L(E)$ and $R(W(x)) \subset \bigcap_{k=1}^{\infty} D\left(Q_\omega(x)^k\right)$. So

$$\begin{gathered} A_\omega(x)\, e^{xQ_\omega(x)}\left(Q_\omega(x) - H\right)^{-1}d_0 + A_\omega(x)\, e^{(1-x)Q_\omega(x)}u_1 = \\ = A_\omega(x)\, e^{xQ_\omega(x)}\left(Q_\omega(x) - H\right)^{-1}d_0 - A_\omega(x)\, e^{xQ_\omega(x)}\left(Q_\omega(0) - H\right)^{-1}d_0 + \\ + A_\omega(x)\, e^{xQ_\omega(x)}\left(Q_\omega(0) - H\right)^{-1}d_0 - A_\omega(0)\, e^{xQ_\omega(0)}\left(Q_\omega(0) - H\right)^{-1}d_0 + \\ + A_\omega(x)\, e^{(1-x)Q_\omega(x)}u_1 - A_\omega(1)\, e^{(1-x)Q_\omega(1)}u_1 + \\ + A_\omega(0)\, e^{xQ_\omega(0)}\left(Q_\omega(0) - H\right)^{-1}d_0 + A_\omega(1)\, e^{(1-x)Q_\omega(1)}u_1 = \\ = -\frac{1}{2\pi i}\int_\Gamma e^{-\sqrt{-z}x}A_\omega(x)\left(A_\omega(x) - z\right)^{-1}\left[\left(Q_\omega(x) - H\right)^{-1} - \left(Q_\omega(0) - H\right)^{-1}\right]d_0dz - \\ -\frac{1}{2\pi i}\int_\Gamma e^{-\sqrt{-z}x}\left[A_\omega(x)\left(A_\omega(x) - z\right)^{-1} - A_\omega(0)\left(A_\omega(0) - z\right)^{-1}\right] \times \left(Q_\omega(0) - H\right)^{-1}d_0dz - \\ -\frac{1}{2\pi i}\int_\Gamma e^{-\sqrt{-z}(1-x)}\left[A_\omega(x)\left(A_\omega(x) - z\right)^{-1} - A_\omega(1)\left(A_\omega(1) - z\right)^{-1}\right]u_1dz + \\ + A_\omega(0)\, e^{xQ_\omega(0)}\left(Q_\omega(0) - H\right)^{-1}d_0 + A_\omega(1)\, e^{(1-x)Q_\omega(1)}u_1. \end{gathered}$$

Using the algebraic identity:

$$\begin{aligned} & A_\omega(x)\left(A_\omega(x) - z\right)^{-1} - A_\omega(0)\left(A_\omega(0) - z\right)^{-1} = \\ = \; & zA_\omega(x)\left(A_\omega(x) - z\right)^{-1}\left[A_\omega(x)^{-1} - A_\omega(0)^{-1}\right]A_\omega(0)\left(A_\omega(0) - z\right)^{-1}, \end{aligned}$$

we obtain:

$$\begin{gathered}
A_\omega(x)\, e^{xQ_\omega(x)} \left(Q_\omega(x) - H\right)^{-1} d_0 + A_\omega(x)\, e^{(1-x)Q_\omega(x)} u_1 = \\
= -\frac{1}{2\pi i} \int_\Gamma e^{-\sqrt{-z}x} A_\omega(x) \left(A_\omega(x) - z\right)^{-1} \left[\left(Q_\omega(x) - H\right)^{-1} - \left(Q_\omega(0) - H\right)^{-1}\right] d_0 dz - \\
-\frac{1}{2\pi i} \int_\Gamma z e^{-\sqrt{-z}x} A_\omega(x) \left(A_\omega(x) - z\right)^{-1} \left[A_\omega(x)^{-1} - A_\omega(0)^{-1}\right] \times \\
\times A_\omega(0) \left(A_\omega(0) - z\right)^{-1} \left(Q_\omega(0) - H\right)^{-1} d_0 dz - \frac{1}{2\pi i} \int_\Gamma z e^{-\sqrt{-z}(1-x)} A_\omega(x) \left(A_\omega(x) - z\right)^{-1} \times \\
\times \left[A_\omega(x)^{-1} - A_\omega(1)^{-1}\right] A_\omega(1) \left(A_\omega(1) - z\right)^{-1} u_1 dz + A_\omega(0)\, e^{xQ_\omega(0)} \left(Q_\omega(0) - H\right)^{-1} d_0 + \\
+ A_\omega(1)\, e^{(1-x)Q_\omega(1)} u_1 = a_1(x) + a_2(x) + a_3(x) + a_4(x) + a_5(x).
\end{gathered}$$

For $a_1(x)$, we have

$$\begin{gathered}
\left\|a_1(x)\right\|_E \leq C \int_\Gamma e^{-c_0|z|^{1/2}x} x^{\alpha+2\mu} |dz| \left\|d_0\right\|_E \leq \\
\leq C \int_0^{+\infty} e^{-\sigma} x^{\alpha+2\mu} \frac{2\sigma d\sigma}{x^2} \left\|d_0\right\|_E \leq C x^{\alpha+2\mu-2} \left\|d_0\right\|_E.
\end{gathered}$$

Then

$$\left(\int_0^1 \left\|a_1(x)\right\|_E^p dx\right)^{\frac{1}{p}} \leq C \left(\int_0^1 x^{(\alpha+2\mu-2)p} dx\right)^{\frac{1}{p}} \left\|d_0\right\|_E < +\infty,$$

so

$$x \longmapsto a_1(x) \in L^p(0,1;E).$$

For the second term, we have:

$$\begin{gathered}
\left\|a_2(x)\right\|_E \leq C \int_\Gamma |z|\, e^{-c_0|z|^{1/2}x} \frac{x^\alpha}{|z|^\mu} \left\|A_\omega(0) \left(A_\omega(0) - z\right)^{-1} \left(Q_\omega(0) - H\right)^{-1} d_0 dz\right\| d|z| \leq \\
\leq C \int_\Gamma |z|\, e^{-c_0|z|^{1/2}x} \frac{x^\alpha}{|z|^\mu} \left\|Q_\omega(0) \left(A_\omega(0) - z\right)^{-1} \left(Q_\omega(0) - H\right)^{-1} Q_\omega(0)\, d_0 dz\right\| d|z| \leq \\
\leq C \int_\Gamma |z|\, e^{-c_0|z|^{1/2}x} \frac{x^\alpha}{|z|^\mu} \left\|\left(A_\omega(0) - z\right)^{-1} Q_\omega(0) \left(Q_\omega(0) - H\right)^{-1} Q_\omega(0)\, d_0 dz\right\| d|z| \leq \\
\leq C \int_\Gamma e^{-c_0|z|^{1/2}x} \frac{x^\alpha}{|z|^\mu} |dz| \left\|Q_\omega(0)\, d_0\right\|_E \leq \\
\leq C \int_0^{+\infty} e^{-\sigma} \frac{x^\alpha}{\left(\dfrac{\sigma^2}{x^2}\right)^\mu} \frac{2\sigma d\sigma}{x^2} \left\|Q_\omega(0)\, d_0\right\|_E \leq C x^{\alpha+2\mu-2} \left\|Q_\omega(0)\, d_0\right\|_E.
\end{gathered}$$

Then

$$\left(\int_0^1 \left\|a_2(x)\right\|_E^p dx\right)^{\frac{1}{p}} \leq C \left(\int_0^1 x^{(\alpha+2\mu-2)p} dx\right)^{\frac{1}{p}} \left\|Q_\omega(0)\, d_0\right\|_E < +\infty,$$

we conclude that:

$$x \longmapsto a_2(x) \in L^p(0,1;E).$$

The same technique is used for the other terms. Finally

$$x \longmapsto G_{Q_\omega(x)}(d_0, u_1, f)(x) \in L^p(0,1;E),$$

if and only if

$$\begin{cases} x \longmapsto A_\omega (0) e^{xQ_\omega(0)} (Q_\omega (0) - H)^{-1} d_0 \in L^p (0, 1; E), \\ x \longmapsto A_\omega (1) e^{(1-x)Q_\omega(1)} u_1 \in L^p (0, 1; E), \end{cases}$$

which is equivalent, by Lemma 8, to

$$\begin{cases} (Q_\omega (0) - H)^{-1} d_0 \in (D (A (0)), E)_{\frac{1}{2p},p}, \\ u_1 \in (D (A (1)), E)_{\frac{1}{2p},p}. \end{cases}$$

□

4.2. Regularity of P_ω

Proposition 3. *Assume (5) – (10). Then for all $\omega \geq \omega_1^*$, we have*

$$P_\omega \in \mathcal{L}\left(L^p (0, 1; E), L^p (0, 1; E) \cap B\left(0, 1; D_{A(\cdot)}\left(\frac{\beta}{2}, +\infty\right)\right)\right),$$

where $\beta \in \left]0, \alpha + 2\mu - 2\right]$

Proof. See [1, Proposition 4, p. 28].

□

Summarizing the above results we obtain the following theorem.

Theorem 1. *Assume (5) – (11). Let $f \in L^p (0, 1; E)$, $1 < p < +\infty$ and*

$$\begin{cases} (Q_\omega (0) - H)^{-1} d_0 \in D (A (0), E)_{\frac{1}{2p},p} \\ u_1 \in D (A (1), E)_{\frac{1}{2p},p}. \end{cases}$$

Then for all $\omega \geq \omega_1^*$, the equation (12) has a unique solution $w(\cdot) = A_\omega(\cdot) u(\cdot)$ verifying:

1) $A_\omega (\cdot) u (\cdot) \in L^p (0, 1; E)$;

2) $u'' \in W^{2,p} (0, 1; E)$.

Proof. We have

$$\begin{aligned} u''(\cdot) = f(\cdot) + A_\omega(\cdot) u(\cdot) = f(\cdot) + \left[G_{Q_\omega(x)} (d_0, u_1, f)(\cdot) - (P_\omega w)(\cdot)\right] = \\ = \left[f(\cdot) + G_{Q_\omega(x)} (d_0, u_1, f)(\cdot)\right] - (P_\omega w)(\cdot). \end{aligned}$$

□

5. The Approximating Problem

In the second section we supposed the existence of a strict solution of problem (1) – (3) and by using a heuristic reasoning we constructed a representation of the solution. Now to prove the existence of the solution, we consider the following approximating problem

$$\begin{cases} u_n''(x) + A_n(x) u_n(x) - \omega u_n(x) = f(x), \quad x \in \left]0, 1\right[, \\ u_n'(0) - H u_n(0) = d_0, \\ u_n(0) = u_1, \end{cases} \tag{14}$$

where $(A_n(x))_{x\in[0,1]}$ is the family of Yosida approximations of $(A(x))_{x\in[0,1]}$ defined by

$$A_n(x) = -nA(x)(A(x)-nI)^{-1}, \ \ n\in\mathbb{N}^*.$$

We use the same arguments as in [20–22], to show that $u_n \to u$.

6. A Concrete General Example

Consider the complex Banach space $E = L^p(\Omega)$, $1 < p < +\infty$ with its usual norm, then E is an UMD Banach space. More precisely, Ω is a regular open of $\mathbb{R}^n$, so we set

$$\begin{cases} E(x,y,D) = \sum\limits_{i,j=1}^{n} a_{ij}(x,y) D_i D_j + \sum\limits_{i=1}^{n} bi(x,y) D_i + c(x,y), & (x,y)\in[0,1]\times\overline{\Omega}, \\ \Gamma(x,s,D) = \sum\limits_{i=1}^{n}\gamma_i(x,s) D_i + \delta(x,s), & (x,s)\in[0,1]\times\partial\Omega, \end{cases}$$

where the coefficients a_{ij}, b_i, c, γ_i and δ are functions defined on $[0,1]\times\overline{\Omega}$ verify the following hypotheses
$\exists\nu > 0 : \forall(x,y)\in[0,1]\times\overline{\Omega}$, $\forall\xi\in\mathbb{R}^n$

$$Re\sum_{i,j=1}^{n} a_{ij}(x,y)\xi_i\xi_j \geq \nu|\xi|^2, \tag{15}$$

$\forall(x,y)\in[0,1]\times\overline{\Omega}$

$$Im\gamma_i(x,y) = 0 \ \text{ and } \ \sum_{i=1}^{n}\gamma_i(x,y)\nu_i(y) \neq 0, \tag{16}$$

where $v(y)$ is the unit outward normal vector to $\partial\Omega$ at y,

$$\begin{cases} a_{ij}(x,.), \, b_i(x,.), \, c(x,.) \in C\left(\overline{\Omega}\right) & \text{uniformly } x\in[0,1], \\ \gamma_i(x,.), \, \delta_i(x,.) \in C^1\left(\overline{\Omega}\right) & \text{uniformly } x\in[0,1], \end{cases} \tag{17}$$

and there exist σ, $K > 0$ such that for all $x\in[0,1]$, $y\in\Omega$ and $s\in\partial\Omega$

$$\begin{cases} \sum\limits_{i,i=1}^{n}|a_{ij}(x_1,y) - a_{ij}(x_2,y)| + \\ +\sum\limits_{i=1}^{n}|b_i(x_1,y) - b_i(x_2,y)| + |c(x_1,y) - c(x_2,y)| \leq K|x_1-x_2|^{\sigma}, \\ \sum\limits_{i=1}^{n}|\gamma_i(x_1,s) - \gamma_i(x_2,s)| + |\delta(x_1,s) - \delta(x_2,s)| \leq K|x_1-x_2|, \\ \sum\limits_{i,k=1}^{n}|D_k\gamma_i(x_1,s)| + \sum\limits_{k=1}^{n}|D_k\delta(x_2,s)| \leq K. \end{cases} \tag{18}$$

Now consider the following concrete problem

$$\begin{cases} \dfrac{\partial^2 u}{\partial x^2}(x,y) + E(x,y,D)u(x,y) - \omega u(x,y) = f(x,y), & (x,y)\in[0,1]\times\overline{\Omega}, \\ \Gamma(x,s,D)u|\partial\Omega = 0, & (x,s)\in[0,1]\times\partial\Omega, \\ \dfrac{\partial u}{\partial x}(0,y) - \alpha u(0,y) = 0, \\ u(1,y) = 0. \end{cases} \tag{19}$$

We define the family of closed linear operators $A(x)$ for all $x \in [0,1]$ by

$$\begin{cases} D(A(x)) = \{u \in W^{2,p}(\Omega) : \Gamma(x,s,D)u = 0, \quad s \in \partial\Omega\}, \\ (A(x)u)(y) = (E(x,y,D))u(y). \end{cases} \tag{20}$$

Let us define the linear operator H by $H = \alpha I$, where $\alpha > 0$.

Then problem (19) is a particular case of problem (1). To apply the results obtained in the previous section we must show that the family $\{A(x)\}_{x\in[0,1]}$ and operator H verify hypotheses (5) – (10).

Proposition 4. *Under hypotheses (15) – (17), the family $\{A(x)\}_{x\in[0,1]}$ defined by (20) verifies hypotheses (6) with $\omega = \omega(p)$, $k = C(p)$, and $\delta = \varphi_0$.*

Proof. This is an immediate consequence of the following result.

□

Theorem 2. *Under hypotheses (15) – (17), there exists $\varphi_0 \in]\pi/2,\pi[$ and $\omega = \omega(p) \geq 0$ such that for every*

$$\forall z \in \sum_{\varphi,\omega} = \{z \in \mathbb{C} : \arg|z-\omega| \leq \varphi_0\},$$

and $x \in [0,1]$, the problem

$$\begin{cases} E(x,.,D)u - zu = f \in L^p(\Omega), \\ \Gamma(x,.,D)u = g \in W^{1-1/p,p}(\partial\Omega), \end{cases}$$

has a unique solution $u(x,.) \in W^{2,p}(\Omega)$. Moreover there exists $C(p) > 0$ such that

$$|z-\omega|\,\|u\|_{L^p(\Omega)} + |z-\omega|^{1/2}\|Du\|_{L^p(\Omega)} + \|D^2u\|_{L^p(\Omega)} \leq$$
$$\leq C(p)\left[\|f\|_{L^p(\Omega)} + \inf_{w\in W^{1,p}(\Omega)}\left(|z-\omega|^{1/2}\|w\|_{L^p(\Omega)} + \|Dw\|_{L^p(\Omega)}\right)\right],$$

where $w = g$ on $\partial\Omega$.

Proof. For the demonstration of this result see in [33, 34].

□

Proposition 5. *We assume $p > n$ and the hypotheses (15) – (18). Then the family $\{A(x)\}_{x\in[0,1]}$ defined by (20) verifies hypotheses (8) where $\omega = \omega(p)$, $(\alpha_1,\mu_1) = (\sigma,1)$, $(\alpha_2,\mu_2) = (1+1/p,1/2)$ and $(\alpha_3,\mu_3) = (1/p,1)$.*

Proof. Let $0 < x_1 < x_2 < 1$, we have

$$(A(x_1)-\omega I)(A(x_1)-zI)^{-1}\left[(A(x_1)-\omega I)^{-1} - (A(x_2)-\omega I)^{-1}\right] =$$
$$= (A(x_1)-\omega I)(A(x_1)-zI)^{-1}\left[(A(x_1)-\omega I)^{-1}(A(x_2)-\omega I) - I\right](A(x_2)-\omega I)^{-1} =$$
$$= (A(x_1)-zI)^{-1}(A(x_2)-\omega I)(A(x_2)-\omega I)^{-1} - (A(x_1)-\omega I)(A(x_1)-zI)^{-1}(A(x_2)-\omega I)^{-1} =$$
$$= (A(x_1)-zI)^{-1}(A(x_2)-zI+zI-\omega I)(A(x_2)-\omega I)^{-1} -$$
$$-(A(x_1)-zI+zI-\omega I)(A(x_1)-zI)^{-1}(A(x_2)-\omega I)^{-1} =$$
$$= (A(x_1)-zI)^{-1}(A(x_2)-zI)(A(x_2)-\omega I)^{-1} - (A(x_1)-zI)(A(x_1)-zI)^{-1}(A(x_2)-\omega I)^{-1} =$$
$$= \left[(A(x_1)-zI)^{-1}(A(x_2)-zI) - I\right](A(x_2)-\omega I)^{-1}.$$

Let $f \in L^p(\Omega)$ and set $v = (A(x_2) - \omega I)^{-1} f,\ u = (A(x_1) - zI)^{-1}(A(x_2) - zI)v$. We must estimate $\|u - v\|_{L^p(\Omega)}$. We have

$$\begin{cases} E(x_2, y, D)v - \omega v = f, & y \in \Omega, \\ \Gamma(x_2, s, D)v = 0, & s \in \partial\Omega, \end{cases} \tag{21}$$

and

$$\begin{cases} E(x_1, y, D)u - zu = E(x_2, y, D)v - zv, & y \in \Omega, \\ \Gamma(x_1, s, D)u = 0, & s \in \partial\Omega, \end{cases}$$

therefore $u - v$ is solution of the following problem

$$\begin{cases} E(x_1, y, D)(u - v) - z(u - v) = E(x_2, y, D)v - E(x_1, y, D)v, & y \in \Omega, \\ \Gamma(x_1, s, D)(u - v) = [\Gamma(x_2, s, D) - \Gamma(x_1, s, D)]v = g & s \in \partial\Omega. \end{cases}$$

Now, let $\Phi_{x_1,x_2}(.) \in D(\Omega)$ satisfy

$$\begin{cases} \Phi_{x_1,x_2}(y) = 1 \text{ if } d(y, \partial\Omega) \leq (x_2 - x_1)/2, \\ \Phi_{x_1,x_2}(y) = 0 \text{ if } d(y, \partial\Omega) \geq (x_2 - x_1), \\ \left| \dfrac{\partial \Phi_{x_1,x_2}}{\partial y_k}(y) \right| \leq \dfrac{C}{x_2 - x_1}, \quad k = 1, 2, ..., n. \end{cases}$$

Applying estimate (2) in the previous Proposition by taking $w = \Phi_{x_1,x_2}(.)g$ (which verifies $w = g$ on $\partial\Omega$ by construction), we then obtain

$$\begin{aligned} |z| \|u - v\|_{L^p(\Omega)} \leq C(p) [\| [E(x_2, y, D) - E(x_1, y, D)] v\|_{L^p(\Omega)} + \\ + |z|^{1/2} \|\Phi_{x_1,x_2}(.)g\|_{L^p(\Omega)} + \|D\Phi_{x_1,x_2}(.)g\|_{L^p(\Omega)}]. \end{aligned} \tag{22}$$

From our assumptions, we have

$$\|[E(x_2, y, D) - E(x_1, y, D)]v\|_{L^p(\Omega)} \leq K(x_2 - x_1)^\sigma \|v\|_{W^{2,p}(\Omega)} \leq K(x_2 - x_1)^\sigma \|f\|_{L^p(\Omega)}.$$

The estimate of the two last terms in (22) needs some technical details in the two cases: $p > n$ and $p \leq n$. The following lemma treats the first case.

□

Lemma 9. *Assume that $p > n$. Set*

$$\Omega_{x_2 - x_1} = \{y \in \Omega : \ d(y, \partial\Omega) < x_2 - x_1\},$$

then we have

$$\int_{\Omega_{x_2 - x_1}} |v(y)|^p \, dy \leq C(x_2 - x_1) \|v\|^p_{W^{2,p}(\Omega)},$$

$$\sum_{i=1}^{n} \int_{\Omega_{x_2 - x_1}} |D_i v(y)|^p \, dy \leq C(x_2 - x_1) \|v\|^p_{W^{2,p}(\Omega)}.$$

Proof. Since $p > n$, we have $W^{1,p}(\Omega) \subset L^\infty(\Omega)$ from which we deduce

$$\sum_{i=1}^{n} \int_{\Omega_{x_2 - x_1}} |D_i v(y)|^p \, dy \leq (x_2 - x_1)(\text{meas } \partial\Omega) \sum_{i=1}^{n} \left(\max_{\Omega} |D_i v(y)| \right)^p \leq C(x_2 - x_1) \|v\|^p_{W^{2,p}(\Omega)}.$$

In the same manner we get the first estimate. Going back to (22), it follows

$$
\begin{gathered}
|z|^{1/2} \left\| \Phi_{x_1,x_2}(.)\, g \right\|_{L^p(\Omega)} \leq |z|^{1/2} \left\| \left[\Gamma(x_2, y, D) - \Gamma(x_1, y, D) \right] v \right\|_{L^p(\Omega)} \leq \\
\leq |z|^{1/2} \left[\sum_{i=1}^{n} \int_{\Omega_{x_2 - x_1}} |\gamma_i(x_2, y) - \gamma_i(x_1, y)|^p \, |D_i v(y)|^p \, dy + \right. \\
\left. + \int_{\Omega_{x_2 - x_1}} |\delta(x_2, y) - \delta(x_1, y)|^p \, |v(y)|^p \, dy \right]^{1/p} \leq \\
\leq K |z|^{1/2} (x_2 - x_1) \left(\sum_{i=1}^{n} \int_{\Omega_{x_2 - x_1}} |D_i v(y)|^p \, dy + \int_{\Omega_{x_2 - x_1}} |v(y)|^p \, dy \right)^{1/p} \leq \\
\leq K |z|^{1/2} (x_2 - x_1)^{1+1/p} \|v\|_{W^{2,p}(\Omega)} \leq K |z|^{1/2} (x_2 - x_1)^{1+1/p} \|f\|_{L^p(\Omega)},
\end{gathered}
$$

and

$$
\begin{gathered}
\left\| D\Phi_{x_1,x_2}(.)\, g \right\|_{L^p(\Omega)} = \left\| D\Phi_{x_1,x_2}(.) \left[\Gamma(x_2, y, D) - \Gamma(x_1, y, D) \right] v \right\|_{L^p(\Omega)} \leq \\
\leq \left\| D \left[\Phi_{x_1,x_2}(.) \sum_{i=1}^{n} \left(\gamma_i(x_2, y) - \gamma_i(x_1, y) \right) D_i v \right] \right\|_{L^p(\Omega)} + \\
+ \left\| D \left[\Phi_{x_1,x_2}(.) \left(\delta(x_2, y) - \delta(x_1, y) \right) v \right] \right\|_{L^p(\Omega)} \leq \\
\leq \sum_{i,k=1}^{n} \left\| D_k \Phi_{x_1,x_2}(.) \left[\left(\gamma_i(x_2, y) - \gamma_i(x_1, y) \right) D_i v \right] \right\|_{L^p(\Omega)} + \\
+ \sum_{k=1}^{n} \left\| D_k \Phi_{x_1,x_2}(.) \left[\left(\delta(x_2, y) - \delta(x_1, y) \right) v \right] \right\|_{L^p(\Omega)} + \\
+ \sum_{i,k=1}^{n} \left\| \Phi_{x_1,x_2}(.)\, D_k \left[\left(\gamma_i(x_2, y) - \gamma_i(x_1, y) \right) D_i v \right] \right\|_{L^p(\Omega)} + \\
+ \sum_{k=1}^{n} \left\| \Phi_{x_1,x_2}(.)\, D_k \left[\left(\delta(x_2, y) - \delta(x_1, y) \right) v \right] \right\|_{L^p(\Omega)} + \\
+ \sum_{i,k=1}^{n} \left\| \Phi_{x_1,x_2}(.) \left[\left(\gamma_i(x_2, y) - \gamma_i(x_1, y) \right) D_k D_i v \right] \right\|_{L^p(\Omega)} + \\
+ \sum_{k=1}^{n} \left\| \Phi_{x_1,x_2}(.) \left[\left(\delta(x_2, y) - \delta(x_1, y) \right) D_k v \right] \right\|_{L^p(\Omega)}.
\end{gathered}
$$

Using our hypotheses and the previous Lemma, we deduce

$$
\begin{gathered}
\left\| D\Phi_{x_1,x_2}(.)\, g \right\|_{L^p(\Omega)} \leq \\
\leq K \left(\frac{C}{x_2 - x_1} (x_2 - x_1) + K \right) \left(\sum_{i=1}^{n} \int_{\Omega_{x_2 - x_1}} |D_i v(y)|^p \, dy + \int_{\Omega_{x_2 - x_1}} |v(y)|^p \, dy \right)^{1/p} \leq \\
\leq K (x_2 - x_1) \|v\|_{W^{2,p}(\Omega)} \leq K (x_2 - x_1)^{1/p} \|v\|_{W^{2,p}(\Omega)} \leq K (x_2 - x_1)^{1/p} \|f\|_{L^p(\Omega)}.
\end{gathered}
$$

Therefore

$$
\|u - v\|_{L^p(\Omega)} \leq \frac{K}{|z|} \left[(x_2 - x_1)^{\sigma} + |z|^{1/2} (x_2 - x_1)^{1+1/p} + (x_2 - x_1)^{1/p} \right] \|f\|_{L^p(\Omega)}.
$$

Then assumption (8) is satisfied with $\omega = \omega(p)$ and $(\alpha_1, \mu_1) = (\sigma, 1)$, $(\alpha_2, \mu_2) = (1 + 1/n, 1/2)$, $(\alpha_3, \mu_3) = (1/n, 1)$ if $p < n$, $(\alpha_1, \mu_1) = (\sigma, 1)$, $(\alpha_2, \mu_2) = (1 + 2r, 1/2)$, $(\alpha_3, \mu_3) = (r, 1)$ if $p = n$ and $r < 1/2n$.

□

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References

1. Haoua R. Differential Equations of Elliptic Type with Variable Operators and General Robin Boundary Condition in UMD Spaces. *Bulletin of the South Ural State University. Series: Mathematical Modeling, Programming*, 2022, vol. 15, no. 4, pp. 20–31. DOI: 10.14529/mmp220402
2. Bourgain J. Some Remarks on Banach Spaces in which Martingale Difference Sequences are Unconditional. *Arkiv for Matematik*, 1983, vol. 21, pp. 163–168. DOI: 10.1007/BF02384306
3. Burkholder D.L. A Geometrical Characterization of Banach Spaces in Which Martingale Difference Sequences are Unconditional. *Annals of Probability*, 1981, vol. 9, pp. 997–1011.
4. Krein S.G. *Linear Differential Equations in Banach Spaces*. Moscow, Nauka, 1967.
5. Balakrishnan A.V. Fractional Powers of Closed Operators and the Semigroups Generated by them. *Pacific Journal of Mathematics*, 1960, vol. 10, pp. 419–437. DOI: 10.2140/PJM.1960.10.419
6. Martinez Carracedo C., Sanz Alix M. *The Theory of Fractional Powers of Operators*. North-Holland Mathematics Studies 187, New York, Elsevier Science, 2001.
7. Carleman T. La Theorie des Equations Integrales et Ses Applications. *Annales de l'institut Henri Poincare*, 1930, vol. 1, no. 4, pp. 401–430. (in French)
8. Bade W.G., Freeman R.S. Closed Extensions of Laplace Operator Determined by a General Class of Boundary Conditions. *Pacific Journal of Mathematics*, 1962, vol. 12, no. 2, pp. 395–410.
9. Beals R. Nonlocal Elliptic Boundary-Value Problems. *Bulletin of the American Mathematical Society*, 1964, vol. 70, no. 5, pp. 693–696.
10. Browder F. Non-Local Elliptic Boundary Value Problems. *American Journal of Mathematics*, 1964, vol. 86, no. 4, pp. 735–750. DOI: 10.2307/2373156
11. Vishik M.J. On General Boundary Value Problems for Elliptic Differential Equations. *American Mathematical Society Transl*, 1963, vol. 2, no. 24, pp. 107–172.
12. Skubachevskii A.L. Nonclassical Boundary Value Problems. *Journal of Mathematical Sciences*, 2008, vol. 155, no. 2, pp. 199–334. DOI: 10.1007/s10958-008-9218-9
13. Bitsadze A.V., Samarskii A.A. On Some General of Linear Elliptic Boundary Value Problems. *Doklady Akademii Nauk*, 1969, vol. 185, no. 10, pp. 739–740.
14. Gurevich P.L. Elliptic Problems with Nonlocal Boundary Conditions and Feller Semigroups. *Journal of Mathematical Sciences*, 2012, vol. 182, no. 3, pp. 255–440. DOI: 10.1007/s10958-012-0746-y
15. Yakubov S., Yakubov Y. *Differential-Operator Equations. Ordinary and Partial Differential Equations*, Boca Raton, Chapman and Hall/CRC, 2000.
16. Aliev B.A., Yakubov Y. Second order Elliptic Differential-Operator Equations with Unbounded Operator Boundary Conditions in UMD Banach Spaces. *Integral Equations and Operator Theory*, 2011, vol. 69, no. 2, pp. 269–300.
17. Favini A., Yakubov Y. Irregular Boundary Value Problems for Second Order Elliptic Differential Operator in UMD Banach Space. *Mathematische Annalen*, 2010, vol. 348, pp. 601–632. DOI: 10.1007/s00208-010-0491-9
18. Cheggag M., Favini A., Labbas R., Maingot S., Medeghri A. Abstract Differential Equations of Elliptic Type wich General Robin Boundary Conditions in Hölder Spaces. *Applicable Analysis*, 2012, vol. 91, no. 8, pp. 1453–1475.

19. Cheggag M., Favini A., Labbas R., Maingot S., Medeghri A. Sturm-Liouville Problems for an Abstract Differential Equation of Elliptic Type in UMD Spaces. *Differential and Integral Equations*, 2008, vol. 21, no. 9-10, pp. 981–1000.

20. Labbas R. *Problèmes aux Limites Pour une Equation Differentielle Abstraite de Type Elliptique.* These d'etat, Universite de Nice, 1987.

21. Bouziani F., Favini A., Labbas R., Medeghri A. Study of Boundary Value and Transmission Problems Governed by a Class of Variable Operators Verifying the Labbas–Terreni non Commutativity Assumption. *Rev Mat Complut*, 2011, vol. 24, pp. 131–168. DOI: 10.1007/s13163-010-0033-8

22. Haoua R., Medeghri A. Robin Boundary Value Problems for Elliptic Operational Differential Equations with Variable Operators. *Electronic Journal of Differential Equations*, 2015, vol. 2015, no. 87, pp. 1–19.

23. G. Da Prato, Grisvard P. Sommes d'Operateurs Lineaires et Equations Differentielles Operationnelles. *Journal de Mathematiques Pures et Appliquees*, 1975, vol. 54, pp. 305–387.

24. Boutaous F., Labbas R., Sadallah B-K. Fractional-Power Approach for Solving Complete Elliptic Abstract Differential Equations with Variable-Operator Coefficients. *Electronic Journal of Differential Equations*, 2012, vol. 2012, no. 05, pp. 1–33.

25. Yagi A. On the Abstract Evolution Equation of Parabolic Type. *Osaka Journal of Mathematics*, 1977, vol. 14, pp. 557–568.

26. Acquistapace P., Terreni B. A Unified Approach to Abstract Linear Nonautonomous Parabolic Equations. *Rendiconti del Seminario Matematico della Università di Padova*, 1987, vol. 78, p. 47–107.

27. Labbas R., Terreni B. Sommes d'Operateurs de Type Elliptique et Parabolique. *Partie. Boll. Un. Math. Ital. 1-B*, 1987, vol. 7, pp. 545–569.

28. Pazy A. *Semigroups of Linear Operators and Applications to Partial Differential Equations.* Heidelberg, Berlin, Tokyo, Springer-Verlag, 1983.

29. Lunardi A. *Analytic Semigroups and Optimal Regularity in Parabolic Problems.* Birkhauser, Basel, 1995.

30. Dore G., Venni. A. On the Closedness of the Sum of two Closed Operators. *Mathematische Zeitschrift*, 1987, vol. 196, pp. 189–201. DOI: 10.1007/BF01163654

31. Triebel H. *Interpolation Theory, Function Spaces, Differential Operators.* North Holland, Amsterdam, 1978.

32. Monniaux S. *Generateur Analytique et Regularite Maimale.* Grade de docteur de l'universite de France-comte, 1995. (in French)

33. Tanabe H. *Equations of Evolution, Monographs and Studies in Mathematics.* Pitman, London-San Francisco-Melbourne, 1979.

34. Agmon S. On the Eigenfunctions and on the Eigenvalues of General Elliptic Boundary Value Problems. *Communications on Pure and Applied Mathematics*, 1962, vol. 15, pp. 119–147. DOI: 10.1002/CPA.3160150203

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ДИФФЕРЕНЦИАЛЬНЫЕ УРАВНЕНИЯ ЭЛЛИПТИЧЕСКОГО ТИПА С ПЕРЕМЕННЫМИ ОПЕРАТОРАМИ И ОБЩИМ ГРАНИЧНЫМ УСЛОВИЕМ РОБИНА В ПРОСТРАНСТВАХ UMD

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В данной работе изучается абстрактное дифференциальное уравнение второго порядка эллиптического типа с переменными операторными коэффициентами и общим граничным условием Робина, которое содержит неограниченный линейный оператор. Исследование проводится в случае, когда второй член принадлежит пространству Соболева и использует знаменитую теорему Доре – Венни. В исследовании не предполагается дифференцируемость резольвентных операторов. Приводятся необходимые и достаточные условия на данные, для того чтобы получить существование, единственность классического решения, которое удовлетворяет свойству максимальной регулярности, полученного в предположении Лаббаса – Террени. Используемые методы по существу основаны на теории полугрупп, дробных степенях линейных операторов, функциональном исчислении Данфорда и теории интерполяции. Работа является продолжением работ, изученных Р. Хауа в пространствах UMD и однородных случаях. Приведен пример, к которому применима данная теория.

Ключевые слова: абстрактные эллиптические дифференциальные уравнения второго порядка; граничные условия Робина; аналитическая полугруппа; максимальная регулярность.

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