МАТЕМАТИЧЕСКОЕ МОДЕЛИРОВАНИЕ

MSC 35K70 DOI: 10.14529/mmp250201

# ON THE MATHEMATICAL MODEL OF THE CONTROL PROBLEM FOR A PSEUDO-PARABOLIC EQUATION WITH INVOLUTION

***F.N. Dekhkonov,*** Namangan State University, Namangan, Uzbekistan,
f.n.dehqonov@mail.ru

In this paper, we consider the mathematical model of control problem for a pseudo-parabolic equation with involution in a bounded two-dimensional domain. The solution with the control function on the border of the considered domain is given. The constraints on the control are determined to ensure that the average value of the solution within the considered domain attains a given value. The initial-boundary problem is solved by the Fourier method, and the control problem under consideration is analyzed with the Volterra integral equation of the second kind. The existence of admissible control is proved by the Laplace transform method.

*Keywords: pseudo-parabolic equation; mathematical model; boundary problem; Volterra integral equation; admissible control; Laplace transform; involution.*

## Introduction

It is known that in recent years, due to the increasing interest in physics and mathematics, the boundary problems related to pseudo-parabolic equations with involution were widely studied. For this purpose, various boundary problems for parabolic and pseudo-parabolic equations have been widely studied by many researchers.

The pseudo-parabolic type equations arise in areas such as fluid flow [1], heat transfer [2], and the diffusion of radiation [3]. Roughly speaking, pseudo-parabolic equations account for higher order correction in the model than do parabolic equations. The boundary control of a pseudo-parabolic equation and compare the results to those of parabolic equations was studied in [4]. The stability, uniqueness, and existence of solutions of some classical problems for the considered equation are studied in [5]. In [6], the point control problems for pseudo-parabolic and parabolic type equations are considered.

The optimal control problem for the parabolic type equations was studied by Fattorini and Friedman [7, 8]. Control problems for the infinite-dimensional case were studied by Egorov [9], who generalized Pontryagin's maximum principle to a class of equations in Banach space, and the proof of a bang-bang principle was shown in the particular conditions.

The boundary control problem for a parabolic equation with a piecewise smooth boundary in a $n$-dimensional domain was studied in [10] and an estimate for the minimum time required to reach a given average temperature was found. In [11], the considered the heat conduction equation with the Robin boundary condition and developed a mathematical model of the process of heating a cylindrical domain. Control problems for the heat transfer equation in the three-dimensional domain are studied in [12].

Control problems for parabolic equations in bounded one and two-dimensional domains are studied in works [13–17]. In these articles, an estimate was found for the minimum time required to heat a bounded domain was found to be an estimate of average temperature. The existence of control function is proved by Laplace transform method.

Basic information on optimal control problems is given in detail in monographs by Lions and Fursikov [18,19]. General numerical optimization and optimal control for second-order parabolic equations have been studied in many publications such as [20]. Practical applications of optimal control problems for equations of parabolic type were presented in [21].

In [22], a boundary value problem for the parabolic equation associated with involution in a one-dimensional domain is studied. Many boundary value problems for parabolic type equations with involution were studied in works [23, 24]. Boundary control problems for pseudo-parabolic type equations were studied in works [25, 26], and it was proved that there is a control function for heating the domain to the average temperature.

In this work, the boundary control problem for the pseudo-parabolic equation with involution is considered. The main control problem is presented in Section 1. In Section 2, the boundary control problem studied in this work is reduced to the Volterra integral equation of the second kind by the Fourier method, and the existence of a solution to the integral equation is proved using the Laplace transform method.

## 1. Statement of Problem

In this paper, we consider the following pseudo-parabolic equation with involution in the domain $\Omega = (0, \pi) \times (0, \pi)$

$$\begin{gathered} u_t(x,y,t) - \Delta u(x,y,t) - \Delta u_t(x,y,t) + \\ +\varepsilon\, u_{xx}(\pi - x, y, t) + \varepsilon u_{xxt}(\pi - x, y, t) \;=\; 0, \\ (x,y,t) \in \Omega_T := \Omega \times (0,T), \end{gathered} \tag{1}$$

with boundary conditions

$$u(0,y,t) \;=\; \psi(y)\,\nu(t), \quad u(\pi,y,t) \;=\; 0, \quad t \geq 0, \quad 0 \leq y \leq \pi, \tag{2}$$

and

$$u(x,0,t) \;=\; 0, \quad u(x,\pi,t) \;=\; 0, \tag{3}$$

and initial condition

$$u(x,y,0) \;=\; 0, \quad 0 \leq x, y \leq \pi, \tag{4}$$

where $\varepsilon$ is a nonzero real number such that $|\varepsilon| < 1$, $\psi(y)$ is a given function, and $\nu(t)$ is the control function. If the control function $\nu(t) \in W_2^1(\mathbb{R}_+)$ satisfies the conditions, $\nu(0) = 0$ and $|\nu(t)| \leq 1$ on the half-line $t \geq 0$, we say that it is admissible.

Assume that the given function $\psi \in W_2^2(\Omega)$ satisfies the conditions

$$\psi(0) \;=\; \psi(\pi) \;=\; 0, \quad \psi_n \geq 0,$$

where

$$\psi_n \;=\; \frac{2}{\pi} \int\limits_0^{\pi} \psi(y)\, \sin ny\, dy, \quad n = 1, 2, \ldots.$$

Differential equations with modified arguments are equations in which the unknown function and its derivatives are evaluated with modifications of time or space variables;

such equations are called, in general, functional differential equations. Among such equations, one can single out, equations with involutions [27].

**Definition 1.** [28,29] *A function $f(x) \not\equiv x$ maps bijectively a set of real numbers $D$, such that*

$$f(f(x)) \;=\; x \quad \textit{or} \quad f^{-1}(x) \;=\; f(x),$$

*is called an involution on $D$.*

It can be seen that equation (1) for $\varepsilon = 0$ is a classical pseudo-parabolic equation. If $\varepsilon \neq 0$, equation (1) relates the values of the second derivatives at two different points and becomes a nonlocal equation. It is known that boundary control problems for the pseudo-parabolic equation in the case $\varepsilon = 0$ were studied in detail in work [26].

We now consider the following control problem.

**Control Problem.** *Assume that function $\phi(t)$ is given. Then, find the control function $\nu(t)$ from the condition*

$$\int\limits_0^\pi \int\limits_0^\pi u(x,y,t)\,dx\,dy = \phi(t), \quad t \geq 0, \tag{5}$$

*where $u(x,y,t)$ is a solution of the problem* (1) – (4) *and it depends on the control function $\nu(t)$.*

## 2. Main Results

In this section, we consider the reduction of the given control problem to a Volterra integral equation of the second kind and the existence of a control function.

We now consider the spectral problem

$$V_{xx}(x,y) - \varepsilon\, V_{xx}(\pi - x, y) + V_{yy}(x,y) + \lambda\, V(x,y) \;=\; 0, \quad 0 < x, y < \pi,$$

$$V(0,y) = V(\pi,y) = 0, \quad V(x,0) = V(x,\pi) = 0, \quad 0 \leq x, y \leq \pi,$$

where $|\varepsilon| < 1$, $\varepsilon \in \mathbb{R} \setminus \{0\}$. It is proved in [22,24] that expressing the solution of a spectral problem in terms of the sum of even and odd functions, one finds the following eigenvalues:

$$\lambda_{m,n,1} = (1-\varepsilon)\,(2m+1)^2 + n^2, \quad m \in \mathbb{N}_0 = \mathbb{N} \cup \{0\}, \quad n \in \mathbb{N}, \tag{6}$$

$$\lambda_{m,n,2} = (1+\varepsilon)\,4\,m^2 + n^2, \quad m, n \in \mathbb{N}, \tag{7}$$

and we have the following eigenfunctions

$$V_{m,n,1}(x,y) = \sin(2m+1)x\,\sin ny, \quad m \in \mathbb{N}_0, \quad n \in \mathbb{N},$$

and

$$V_{m,n,2}(x,y) = \sin 2mx\,\sin ny, \quad m, n \in \mathbb{N}.$$

By the solution of the problem (1) – (4) we understand the function $u(x,y,t)$ represented in the form

$$u(x,y,t) = \frac{\pi - x}{\pi}\,\psi(y)\,\nu(t) - w(x,y,t), \tag{8}$$

where the function $w(x,y,t) \in C^{2,2,1}_{x,y,t}(\Omega_T) \cap C(\bar{\Omega}_T)$ is the solution to the problem:

$$w_t(x,y,t) - \Delta w(x,y,t) - \Delta w_t(x,y,t) + \varepsilon\, w_{xx}(\pi - x, y, t) + \varepsilon w_{xxt}(\pi - x, y, t) =$$
$$= \frac{\pi - x}{\pi}\,\psi(y)\,\nu'(t) - \frac{\pi - x}{\pi}\,\psi''(y)\,\nu(t) - \frac{\pi - x}{\pi}\,\psi''(y)\,\nu'(t)$$

with initial-boundary conditions

$$w(x,y,t)\mid_{\partial\Omega} = 0, \quad w(x,y,0) = 0.$$

We obtain the solution of the initial boundary value problem (1) – (4) using the solution of the above mixed problem $w(x,y,t)$ and the equality (8) as follows:

$$\begin{gathered} u(x,y,t) = \frac{\pi - x}{\pi}\,\psi(y)\,\nu(t) - \\ -\frac{2}{\pi}\sum_{m=0}^{\infty}\sum_{n=1}^{\infty}\frac{(1+n^2)\,\psi_n\,V_{m,n,1}(x,y)}{(2m+1)\,(1+\lambda_{m,n,1})}\int_0^t e^{-q_{m,n,1}(t-s)}\,\nu'(s)\,ds - \\ -\frac{2}{\pi}\sum_{m=0}^{\infty}\sum_{n=1}^{\infty}\frac{n^2\,\psi_n\,V_{m,n,1}(x,y)}{(2m+1)\,(1+\lambda_{m,n,1})}\int_0^t e^{-q_{m,n,1}(t-s)}\,\nu(s)\,ds - \\ -\frac{1}{\pi}\sum_{m=1}^{\infty}\sum_{n=1}^{\infty}\frac{(1+n^2)\,\psi_n\,V_{m,n,2}(x,y)}{m\,(1+\lambda_{m,n,2})}\int_0^t e^{-q_{m,n,2}(t-s)}\,\nu'(s)\,ds - \\ -\frac{1}{\pi}\sum_{m=1}^{\infty}\sum_{n=1}^{\infty}\frac{n^2\,\psi_n\,V_{m,n,2}(x,y)}{m\,(1+\lambda_{m,n,2})}\int_0^t e^{-q_{m,n,2}(t-s)}\,\nu(s)\,ds, \end{gathered} \tag{9}$$

where $V_{m,n,i}(x,y)$ are the eigenfunctions, $q_{m,n,i} = \dfrac{\lambda_{m,n,i}}{1+\lambda_{m,n,i}}$ $(i = 1,2)$ and $\lambda_{m,n,1}$, $\lambda_{m,n,2}$ are defined by (6), (7), respectively.

It is clear that

$$\int_0^\pi\int_0^\pi V_{m,n,1}(x,y)\,dx\,dy = \frac{2\,(1-(-1)^n)}{n\,(2m+1)}, \quad \int_0^\pi\int_0^\pi V_{m,n,2}(x,y)\,dx\,dy = 0. \tag{10}$$

Using the condition (5) and the equalities (9), (10), we have

$$\begin{gathered} \phi(t) = \int_0^\pi\int_0^\pi u(x,y,t)\,dy\,dx = \nu(t)\int_0^\pi\int_0^\pi \frac{\pi - x}{\pi}\,\psi(y)\,dy\,dx - \\ -\frac{4}{\pi}\sum_{m=0}^{\infty}\sum_{n=1}^{\infty}\frac{(1+n^2)\,\psi_n\,(1-(-1)^n)}{n\,(2m+1)^2\,(1+\lambda_{m,n,1})}\int_0^t e^{-q_{m,n,1}(t-s)}\,\nu'(s)\,ds - \\ -\frac{4}{\pi}\sum_{m=0}^{\infty}\sum_{n=1}^{\infty}\frac{n\,\psi_n\,(1-(-1)^n)}{(2m+1)^2\,(1+\lambda_{m,n,1})}\int_0^t e^{-q_{m,n,1}(t-s)}\,\nu(s)\,ds. \end{gathered} \tag{11}$$

According to Parseval equality, we get

$$\int_0^\pi \int_0^\pi \frac{\pi - x}{\pi}\,\psi(y)\,dy\,dx \;=\; \frac{4}{\pi}\sum_{m=0}^{\infty}\sum_{n=1}^{\infty}\frac{\psi_n\left(1-(-1)^n\right)}{(2m+1)^2\,n}.$$

Set

$$\alpha \;=\; \frac{4\,(1-\varepsilon)}{\pi}\sum_{m=0}^{\infty}\sum_{n=1}^{\infty}\frac{\psi_n\,(1-(-1)^n)}{n\,(1+\lambda_{m,n,1})}, \tag{12}$$

and

$$\Lambda_{m,n} \;=\; \frac{4\,(1-\varepsilon)}{\pi}\frac{\psi_n\,(1-(-1)^n)}{n\,(1+\lambda_{m,n,1})^2}. \tag{13}$$

By (11) – (13) and from the properties of the function $\nu(t)$, we can write

$$\phi(t) = \alpha\,\nu(t) + \sum_{m=0}^{\infty}\sum_{n=1}^{\infty}\Lambda_{m,n}\int_0^t e^{-q_{m,n,1}(t-s)}\,\nu(s)\,ds, \tag{14}$$

where $q_{m,n,1} = \dfrac{\lambda_{m,n,1}}{1+\lambda_{m,n,1}}$.

Consider the function

$$K(t) \;=\; \sum_{m=0}^{\infty}\sum_{n=1}^{\infty}\Lambda_{m,n}\,e^{-q_{m,n,1}\,t}, \quad t>0. \tag{15}$$

Then, using (14), (15), we get the following Volterra integral equation

$$\alpha\,\nu(t) + \int_0^t K(t-s)\,\nu(s)ds = \phi(t), \quad t>0. \tag{16}$$

For any $M>0$, we denote $W(M)$ the set of functions $\phi \in W_2^1(-\infty,+\infty)$ which satisfy the following conditions:

$$\|\phi\|_{W_2^1(\mathbb{R}_+)} \le M, \quad \phi(t)=0 \quad \text{for} \quad t\le 0.$$

Below, we present the main result in this paper.

**Theorem.** (Existence of the solution) *There exists $M>0$ such that for any function $\phi \in W(M)$ the solution $\nu(t)$ of the equation* (16) *exists, and satisfies condition $|\nu(t)| \le 1$.*

We will prove this theorem step by step.

**Lemma 1.** *Assume that, $|\varepsilon|<1$. Then, the kernel $K(t)$ of the integral equation* (16) *is continuous on the half-line $t \ge 0$.*

*Proof.* Since $\psi_n$ is non-negative and $|\varepsilon|<1$, it is clear that $\Lambda_{m,n}$ is non-negative. It is easy to see that then the kernel of the integral equation (16) is the function $K(t)$, which is positive and bounded. □

It is known that we can write the Laplace transform of the function $\nu(t)$ as follows

$$\widetilde{\nu}(p) = \int_0^\infty e^{-pt}\, \nu(t)\, dt, \tag{17}$$

where $p = \sigma + i\zeta$, $\sigma > 0$, $\zeta \in \mathbb{R}$.

Then, applying the Laplace transform to the integral equation (16), we get

$$\widetilde{\phi}(p) = \alpha\, \widetilde{\nu}(p) + \widetilde{K}(p)\, \widetilde{\nu}(p),$$

where $\alpha$ is defined by (12). Then we can write

$$\widetilde{\nu}(p) = \frac{\widetilde{\phi}(p)}{\alpha + \widetilde{K}(p)},$$

and

$$\nu(t) = \frac{1}{2\pi i} \int_{\sigma - i\infty}^{\sigma + i\infty} \frac{\widetilde{\phi}(p)}{\alpha + \widetilde{K}(p)}\, e^{pt} dp = \frac{1}{2\pi} \int_{-\infty}^{+\infty} \frac{\widetilde{\phi}(\sigma + i\zeta)}{\alpha + \widetilde{K}(\sigma + i\zeta)}\, e^{(\sigma + i\zeta)t}\, d\zeta. \tag{18}$$

**Lemma 2.** *The following estimate is valid:*

$$|\alpha + \widetilde{K}(\sigma + i\,\zeta)| \geq \alpha, \quad \sigma > 0, \quad \zeta \in \mathbb{R},$$

*where* $\alpha = const > 0$ *is defined by* (12).

*Proof.* It is clear that $\alpha$ defined by (12) is positive and bounded as $|\varepsilon| < 1$.

It is known that we can write the Laplace transform of the function $K(t)$ as follows

$$\widetilde{K}(p) = \sum_{m=0}^{\infty} \sum_{n=1}^{\infty} \frac{\Lambda_{m,n}}{p + q_{m,n,1}}.$$

Then we can write

$$\alpha + \widetilde{K}(\sigma + i\zeta) = \alpha + \sum_{m=0}^{\infty} \sum_{n=1}^{\infty} \frac{\Lambda_{m,n}}{\sigma + q_{m,n,1} + i\zeta} =$$
$$= \alpha + \sum_{m=0}^{\infty} \sum_{n=1}^{\infty} \frac{\Lambda_{m,n}\,(\sigma + q_{m,n,1})}{(\sigma + q_{m,n,1})^2 + \zeta^2} - i\zeta \sum_{m=0}^{\infty} \sum_{n=1}^{\infty} \frac{\Lambda_{m,n}}{(\sigma + q_{m,n,1})^2 + \zeta^2} =$$
$$= \mathrm{Re}(\alpha + \widetilde{K}(\sigma + i\zeta)) + i\, \mathrm{Im}(\alpha + \widetilde{K}(\sigma + i\zeta)),$$

where

$$\mathrm{Re}(\alpha + \widetilde{K}(\sigma + i\zeta)) = \alpha + \sum_{m=0}^{\infty} \sum_{n=1}^{\infty} \frac{\Lambda_{m,n}\,(\sigma + q_{m,n,1})}{(\sigma + q_{m,n,1})^2 + \zeta^2},$$

and

$$\mathrm{Im}(\alpha + \widetilde{K}(\sigma + i\zeta)) = -\zeta \sum_{m=0}^{\infty} \sum_{n=1}^{\infty} \frac{\Lambda_{m,n}}{(\sigma + q_{m,n,1})^2 + \zeta^2}.$$

We can see that the following inequality holds

$$(\sigma + q_{m,n,1})^2 + \zeta^2 \le ((\sigma + q_{m,n,1})^2 + 1)(1 + \zeta^2).$$

As a result, we get

$$\frac{1}{(\sigma + q_{m,n,1})^2 + \zeta^2} \ge \frac{1}{1+\zeta^2}\,\frac{1}{(\sigma + q_{m,n,1})^2 + 1}. \tag{19}$$

Then, using the inequality (19), we can obtain the following assumptions

$$\begin{gathered} |\mathrm{Re}(\alpha + \widetilde{K}(\sigma + i\zeta))| = \alpha + \sum_{m=0}^{\infty}\sum_{n=1}^{\infty} \frac{\Lambda_{m,n}\,(\sigma + q_{m,n,1})}{(\sigma + q_{m,n,1})^2 + \zeta^2} \ge \\ \ge \alpha + \frac{1}{1+\zeta^2}\sum_{m=0}^{\infty}\sum_{n=1}^{\infty} \frac{\Lambda_{m,n}\,(\sigma + q_{m,n,1})}{(\sigma + q_{m,n,1})^2 + 1} = \alpha + \frac{C_{1,\sigma}}{1+\zeta^2}. \end{gathered} \tag{20}$$

From the above, we can see that the following estimate is valid for $|\mathrm{Im}(\alpha + \widetilde{K}(\sigma + i\zeta))|$:

$$|\mathrm{Im}(\alpha + \widetilde{K}(\sigma + i\zeta))| \ge \frac{C_{2,\sigma}\,|\zeta|}{1+\zeta^2}, \tag{21}$$

where $C_{1,\sigma}$ and $C_{2,\sigma}$ are defined as follows

$$C_{1,\sigma} = \sum_{m=0}^{\infty}\sum_{n=1}^{\infty} \frac{\Lambda_{m,n}\,(\sigma + q_{m,n,1})}{(\sigma + q_{m,n,1})^2 + 1}, \quad C_{2,\sigma} = \sum_{m=0}^{\infty}\sum_{n=1}^{\infty} \frac{\Lambda_{m,n}}{(\sigma + q_{m,n,1})^2 + 1}.$$

By (20) and (21), we obtain the required estimate

$$|\alpha + \widetilde{K}(\sigma + i\zeta)| \ge \alpha + \frac{C_\sigma}{\sqrt{1+\zeta^2}} \ge \alpha, \tag{22}$$

where $C_\sigma = \min(C_{1,\sigma}, C_{2,\sigma})$ is bounded for all $\sigma > 0$.

□

Let the Laplace transform of function $\phi(t)$ satisfy the condition

$$\int_{-\infty}^{+\infty} |\widetilde{\phi}(i\zeta)|\,d\zeta < +\infty.$$

If we proceed to the limit as $\sigma \to 0$ in the equality (18), we have

$$\nu(t) = \frac{1}{2\pi}\int_{-\infty}^{+\infty} \frac{\widetilde{\phi}(i\zeta)}{\alpha + \widetilde{K}(i\zeta)}\,e^{i\zeta t}d\zeta. \tag{23}$$

Also, to prove the theorem, we need the following lemma.

**Lemma 3.** *Assume that, $\phi(t) \in W(M)$. Then for the imaginary part of the Laplace transform of function $\phi(t)$ the following estimate holds*

$$\int_{-\infty}^{+\infty} |\widetilde{\phi}(i\zeta)|\,d\zeta < +\infty.$$

*Proof.* Using the formula for integration by parts in (17), we take

$$(\sigma + i\zeta)\,\widetilde{\phi}(\sigma + i\zeta) = \int\limits_0^\infty e^{-(\sigma+i\zeta)t}\,\phi'(t)\,dt.$$

Further for $\sigma \to 0$ we get

$$i\zeta\,\widetilde{\phi}(i\zeta) = \int\limits_0^\infty e^{-i\zeta t}\,\phi'(t)\,dt.$$

Besides

$$\widetilde{\phi}(i\zeta) = \int\limits_0^\infty e^{-i\zeta t}\,\phi(t)\,dt.$$

Thus, we can write the following inequality:

$$\int\limits_{-\infty}^{+\infty} |\widetilde{\phi}(i\zeta)|^2(1+\zeta^2)\,d\zeta \le C_2\,\|\phi\|^2_{W_2^1(R_+)},$$

where $C_2 = \text{const} > 0$.

By using Cauchy–Bunyakowskii inequalities, we have

$$\begin{aligned}\int\limits_{-\infty}^{+\infty} |\widetilde{\phi}(i\zeta)|\,d\zeta &= \int\limits_{-\infty}^{+\infty} \frac{|\widetilde{\phi}(i\zeta)|}{1+\zeta^2}\,d\zeta + \int\limits_{-\infty}^{+\infty} \frac{\zeta^2\,|\widetilde{\phi}(i\zeta)|}{1+\zeta^2}\,d\zeta \le \\ &\le \Big(\int\limits_{-\infty}^{+\infty} |\widetilde{\phi}(i\zeta)|^2\,d\zeta\Big)^{1/2} \Big(\int\limits_{-\infty}^{+\infty} \frac{1}{(1+\zeta^2)^2}\,d\zeta\Big)^{1/2} + \\ &+ \Big(\int\limits_{-\infty}^{+\infty} \zeta^2\,|\widetilde{\phi}(i\zeta)|^2\,d\zeta\Big)^{1/2} \Big(\int\limits_{-\infty}^{+\infty} \frac{\zeta^2}{(1+\zeta^2)^2}\,d\zeta\Big)^{1/2} \le \\ &\le C \int\limits_{-\infty}^{+\infty} |\widetilde{\phi}(i\zeta)|^2(1+\zeta^2)\,d\zeta \le C_2\,\|\phi\|^2_{W_2^1(\mathbb{R}_+)}.\end{aligned} \tag{24}$$

□

Now we present the proof of the theorem.

**Proof of the Theorem.** Let us show that $\nu \in W_2^1(\mathbb{R}_+)$. Indeed, using (22) and (23) we can write

$$\begin{aligned}\int\limits_{-\infty}^{+\infty} |\widetilde{\nu}(\zeta)|^2(1+|\zeta|^2)\,d\zeta \;&=\; \int\limits_{-\infty}^{+\infty} \left|\frac{\widetilde{\phi}(i\zeta)}{\alpha + \widetilde{K}(i\zeta)}\right|^2 (1+|\zeta|^2)\,d\zeta \;\le \\ &\le \frac{1}{\alpha^2} \int\limits_{-\infty}^{+\infty} |\widetilde{\phi}(i\zeta)|^2(1+|\zeta|^2)\,d\zeta \;=\; \text{const}\,\|\phi\|^2_{W_2^1(\mathbb{R})}.\end{aligned}$$

Now, we show that the function $\nu(t)$ satisfies the Lipschitz condition. Actually,

$$|\nu(t)-\nu(s)| \ = \ \left|\int\limits_s^t \nu'(\xi)\,d\xi\right| \leq \ \|\nu'\|_{L^2(\Omega)}\sqrt{t-s}.$$

Using (22), (23) and Lemma 3, we have the following estimate

$$|\nu(t)| \leq \frac{1}{2\pi}\int\limits_{-\infty}^{+\infty}\frac{|\widetilde{\phi}(i\zeta)|}{|\alpha+\widetilde{K}(i\zeta)|}d\zeta \leq \frac{1}{2\pi\,\alpha}\int\limits_{-\infty}^{+\infty}|\widetilde{\phi}(i\zeta)|\,d\zeta \leq$$
$$\leq \frac{C_2}{2\pi\,\alpha}\|\phi\|_{W_2^1(\mathbb{R}_+)} \leq \frac{C_2\,M}{2\pi\,\alpha} = 1,$$

where $\alpha$ is defined by (12) and

$$M = \frac{2\pi\,\alpha}{C_2}, \quad C_2 = \text{const} > 0.$$

The theorem is proved. $\square$

## Conclusion

In this paper, the control problem for a pseudoparabolic equation involving involution in a bounded two-dimensional domain was considered. Using the Laplace transform method, the control function required to heat the given domain to an average temperature is found, and the admissibility of this control function was proven.

## References


1. Coleman B.D., Noll W. An Approximation Theorem for Functionals, with Applications in Continuum Mechanics. *Archive for Rational Mechanics and Analysis*, 1960, vol. 6, pp. 355–370. DOI: 10.1007/BF00276168
2. Chen Peter J., Gurtin M. On a Theory of Heat Conduction Involving Two Temperatures. *Zeitschrift fur angewandte mathematik und physik*, 1968, vol. 19, pp. 614–627. DOI: 10.1007/BF01594969 (in German)
3. Milne E.A. The Diffusion of Imprisoned Radiation Through a Gas. *Journal of the London Mathematical Society*, 1926, vol. 1, no. 1, pp. 40–51. DOI: 10.1112/jlms/s1-1.1.40
4. White L.W. Controllability Properties of Pseudo-Parabolic Boundary Control Problems. *SIAM Journal on Control and Optimization*, 1980, vol. 18, no. 5, pp. 534–539. DOI: 10.1137/0318039
5. Coleman B.D., Duffin R.J., Mizel V.J. Instability, Uniqueness, and Nonexistence Theorems for the Equation on a Strip. *Archive for Rational Mechanics and Analysis*, 1965, vol. 19, pp. 100–116. DOI: 10.1007/BF00282277
6. White L.W. Point Control of Pseudoparabolic Problems. *Journal of Differential Equations*, 1981, vol. 42, pp. 366–374. DOI: 10.1016/0022-0396(81)90110-8
7. Fattorini H.O. Time-Optimal Control of Solutions of Operational Differential Equation. *Journal of the Society for Industrial and Applied Mathematics. Series A Control*, 1964, vol. 2, no. 1, pp. 49–65. DOI: 10.1137/0302005

8. Friedman A. *Partial Differential Equations of Parabolic Type*. New York, Dover Publications, 1964. DOI: 10.1016/0022-247X(67)90040-6
9. Egorov Yu.V. Optimal Control in Banach Spaces. *Doklady Akademii nauk SSSR*, 1963, vol. 150, no. 2, pp. 241–244. (in Russian)
10. Albeverio S., Alimov Sh.A. On One Time-Optimal Control Problem Associated with the Heat Exchange Process. *Applied Mathematics and Optimization*, 2008, vol. 57, no. 1, pp. 58–68. DOI: 10.1007/s00245-007-9008-7
11. Alimov Sh.A., Komilov N.M. Determining the Thermal Mode Setting Parameters Based on Output Data. *Partial Differential Equations*, 2022, vol. 58, no. 1, pp. 21–35.
12. Dekhkonov F.N. On the Control Problem Associated with the Heating Process. *Mathematical Notes of NEFU*, 2022, vol. 29, no. 4, pp. 62–71. DOI: 10.25587/SVFU.2023.82.41.005
13. Chen Ning, Wang Yanqing, Yang Dong-Hui. Time-Varying Bang-Bang Property of Time Optimal Controls for Heat Equation and Its Applications. *Systems and Control Letters*, 2018, vol. 112, pp. 18–23. DOI: 10.1016/j.sysconle.2017.12.008
14. Fayazova Z.K. Boundary Control of the Heat Transfer Process in the Space. *Izvestiya VUZ. Matematika*, 2019, vol. 63, pp. 71–79. DOI: 10.3103/S1066369X19120089 (in Russian)
15. Dekhkonov F.N. Boundary Control Problem for the Heat Transfer Equation Associated with Heating Process of a Rod. *Bulletin of the Karaganda University. Mathematics Series*, 2023, vol. 110, no. 2, pp. 63–71. DOI: 10.31489/2023m2/63-71
16. Dekhkonov F.N. Boundary Control Associated with a Parabolic Equation. *Journal of Mathematics and Computer Science*, 2024, vol. 33, no. 2, pp. 146–154. DOI: 10.22436/jmcs.033.02.03
17. Dekhkonov F.N., Kuchkorov E.I. On the Time-Optimal Control Problem Associated with the Heating Process of a Thin Rod. *Lobachevskii Journal of Mathematics*, 2023, vol. 44, no. 3, pp. 1134–1144. DOI: 10.1134/S1995080223030101
18. Lions J.L. *Contróle optimal de systémes gouvernés par des équations aux dérivées partielles*. Paris, Dunod Gauthier-Villars, 1968. (in French)
19. Fursikov A.V. *Optimal Control of Distributed Systems, Theory and Applications*. Vol. 187. American Mathematical Society, Providence, 2000.
20. Altmüller A., Grüne L. *Distributed and Boundary Model Predictive Control for the Heat Equation*. University of Bayreuth, Department of Mathematics, 2012, vol. 35, no. 2, pp. 131–145. DOI: 10.1002/gamm.201210010
21. Dubljevic S., Christofides P.D. Predictive Control of Parabolic PDEs with Boundary Control Actuation. *Chemical Engineering Science*, 2006, vol. 61, no. 18, pp. 6239–6248. DOI: 10.1016/j.ces.2006.05.041
22. Mussirepova E., Sarsenbi A., Sarsenbi A. The Inverse Problem for the Heat Equation with Reflection of the Argument and with a Complex Coefficient. *Bound Value Problems*, 2022, vol. 1, article ID: 99, 13 p. DOI: 10.1186/s13661-022-01675-1
23. Kopzhassarova A., Sarsenbi A. Basis Properties of Eigenfunctions of Second-Order Differential Operators with Involution. *Abstract and Applied Analysis*, 2012, article ID: 576843, 6 p. DOI: 10.1155/2012/576843
24. Ahmad B., Alsaedi A., Kirane M., Tapdigoglu R. An Inverse Problem for Space and Time Fractional Evolution Equations with an Involution Perturbation. *Quaestiones Mathematicae*, 2017, vol. 40, no. 2, pp. 151–160. DOI: 10.2989/16073606.2017.1283370 (in French)

25. Dekhkonov F.N. On the Control Problem Associated with a Pseudo-Parabolic Type Equation in an One-Dimensional Domain. *International Journal of Applied Mathematics*, 2024, vol. 37, no. 1, pp. 109–118. DOI: 10.12732/ijam.v37i1.9
26. Dekhkonov F.N. On a Boundary Control Problem for a Pseudo-Parabolic Equation. *Communications in Analysis and Mechanics*, 2023, vol. 15, no. 2, pp. 289–299. DOI: 10.3934/cam.2023015
27. Cabada A., Tojo F.A.F. *General Results for Differential Equations with Involutions.* Paris, Atlantis Press, 2015, pp. 17–23. DOI: 10.2991/978-94-6239-121-5-2
28. Carleman T. Sur la theorie des equations integrales et ses applications. *Verhandlungen des internationalen mathematiker-kongresses*, 1932, pp. 138–151. (in French)
29. Wiener J. *Generalized Solutions of Functional-Differential Equations.* Singapore, World Scientic Publishing, 1999.


*Received August 18, 2024*

**MSC 35K70** **DOI: 10.14529/mmp250201**

## О МАТЕМАТИЧЕСКОЙ МОДЕЛИ ЗАДАЧИ УПРАВЛЕНИЯ ДЛЯ ПСЕВДОПАРАБОЛИЧЕСКОГО УРАВНЕНИЯ С ИНВОЛЮЦИЕЙ

***Ф.Н. Дехконов,*** Наманганский государственный университет, г. Наманган, Узбекистан

В данной работе рассматривается математическая модель задачи управления для псевдопараболического уравнения с инволюцией в ограниченной двумерной области. Приводится решение с функцией управления на границе рассматриваемой области. Ограничения на управление определяются таким образом, чтобы среднее значение решения внутри рассматриваемой области достигало заданного значения. Начально-краевая задача решается методом Фурье, а рассматриваемая задача управления анализируется с помощью интегрального уравнения Вольтерра второго рода. Существование допустимого управления доказывается методом преобразования Лапласа.

*Ключевые слова: псевдопараболическое уравнение; математическая модель; краевая задача; интегральное уравнение Вольтерра; допустимое управление; преобразование Лапласа; инволюция.*

## Литература


1. Coleman, B.D. An Approximation Theorem for Functionals, with Applications in Continuum Mechanics / B.D. Coleman, W. Noll // Archive for Rational Mechanics and Analysis. – 1960. – V. 6. – P. 355–370.
2. Chen, P.J. On a Theory of Heat Conduction Involving Two Temperatures / P.J. Chen, M. Gurtin // Zeitschrift fur angewandte mathematik und physik. – 1968. – V. 19. – P. 614–627.
3. Milne, E.A. The Diffusion of Imprisoned Radiation Through a Gas / E.A. Milne // Journal of the London Mathematical Society. – 1926. – V. 1, № 1. – P. 40–51.

4. White, L.W. Controllability Properties of Pseudo-Parabolic Boundary Control Problems / L.W. White // SIAM Journal on Control and Optimization. – 1980. – V. 18, № 5. – P. 534–539.
5. Coleman, B.D. Instability, Uniqueness, and Nonexistence Theorems for the Equation on a Strip / B.D. Coleman, R.J. Duffin, V.J. Mizel // Archive for Rational Mechanics and Analysis. – 1965. – V. 19. – P. 100–116.
6. White, L.W. Point Control of Pseudoparabolic Problems / L.W. White // Journal of Differential Equations. – 1981. – V. 42 – P. 366–374.
7. Fattorini, H.O. Time-Optimal Control of Solutions of Operational Differential Equation / H.O. Fattorini // Journal of the Society for Industrial and Applied Mathematics Series A Control. – 1964. – V. 2, № 1. – P. 49–65.
8. Friedman, A. Partial Differential Equations of Parabolic Type / A. Friedman. – New York: Dover Publications, 1964.
9. Егоров, Ю.В. Оптимальное управление в банаховом пространстве / Ю.В. Егоров // Доклады Академии наук СССР. – 1963. – Т. 150, № 2. – С. 241–244.
10. Albeverio, S. On One Time-Optimal Control Problem Associated with the Heat Exchange Process / S. Albeverio, Sh.A. Alimov // Applied Mathematics and Optimization. – 2008. – V. 57, № 1. – P. 58–68.
11. Alimov, Sh.A. Determining the Thermal Mode Setting Parameters Based on Output Data / Sh.A. Alimov, N.M. Komilov // Partial Differential Equations. – 2022. – V. 58, № 1. – P. 21–35.
12. Дехконов, Ф.Н. О проблеме управления, связанной с процессом нагрева / Ф.Н. Дехконов // Математические заметки СВФУ. – 2022. – V. 29, № 4.– P. 62–71.
13. Chen Ning. Time-Varying Bang-Bang Property of Time Optimal Controls for Heat Equation and Its Applications / Chen Ning, Yanqing Wang, Dong-Hui Yang // Systems and Control Letters. – 2018. – V. 112. – P. 18–23.
14. Фаязова, З.К. Граничное управление процессом теплообмена в пространстве / З.К. Фаязова // Известия вузов. Математика. – 2019. – Т. 63, № 12. – С. 82–90.
15. Dekhkonov, F.N. Boundary Control Problem for the Heat Transfer Equation Associated with Heating Process of a Rod / F.N. Dekhkonov // Bulletin of the Karaganda University. Mathematics Series. – 2023. – V. 110, № 2. – P. 63–71.
16. Dekhkonov, F.N. Boundary Control Associated with a Parabolic Equation / F.N. Dekhkonov // Journal of Mathematics and Computer Science. – 2024. – V. 33, № 2. – P. 146–154.
17. Dekhkonov, F.N. On the Time-Optimal Control Problem Associated with the Heating Process of a Thin Rod / F.N. Dekhkonov, E.I. Kuchkorov // Lobachevskii Journal of Mathematics. – 2023. – V. 44, № 3. – P. 1134–1144.
18. Lions, J.L. Contróle optimal de systémes gouvernés par des équations aux dérivées partielles / J.L. Lions. – Paris: Dunod Gauthier-Villars, 1968.
19. Фурсиков, А.В. Оптимальное управление распределенными системами. Теория и приложения / А.В. Фурсиков. – Новосибирск: Научная книга, 1999.
20. Altmüller A. Distributed and Boundary Model Predictive Control for the Heat Equation / A. Altmüller, L. Grüne // University of Bayreuth. – 2012. – V. 35, № 2. – P. 131–145.
21. Dubljevic, S. Predictive Control of Parabolic PDEs with Boundary Control Actuation / S. Dubljevic, P.D. Christofides // Chemical Engineering Science. – 2006. – V. 61, № 18. – P. 6239–6248.

22. Mussirepova, E. The Inverse Problem for the Heat Equation with Reflection of the Argument and with a Complex Coefficient / E. Mussirepova, A. Sarsenbi, A. Sarsenbi // Bound Value Problems. – 2022. – V. 1. – Article ID: 99. – 13 p.
23. Kopzhassarova, A. Basis Properties of Eigenfunctions of Second-Order Differential Operators with Involution / A. Kopzhassarova, A. Sarsenbi // Abstract and Applied Analysis. – 2012. – Article ID: 576843. – 6 p.
24. Ahmad, B. An Inverse Problem for Space and Time Fractional Evolution Equations with an Involution Perturbation / B. Ahmad, A. Alsaedi, M. Kirane, R. Tapdigoglu // Quaestiones Mathematicae. – 2017. – V. 40, № 2. – P. 151–160.
25. Dekhkonov, F.N. On the Control Problem Associated with a Pseudo-Parabolic Type Equation in an One-Dimensional Domain / F.N. Dekhkonov // International Journal of Applied Mathematics. – 2024. – V. 37, № 1. – P. 109–118.
26. Dekhkonov, F.N. On a Boundary Control Problem for a Pseudo-Parabolic Equation / F.N. Dekhkonov // Communications in Analysis and Mechanics. – 2023. – V. 15, № 2. – P. 289–299.
27. Cabada, A. General Results for Differential Equations with Involutions / A. Cabada, F.A.F. Tojo // Differential Equations with Involutions. – Paris: Atlantis Press, 2015. – P. 17–23.
28. Carleman, T. Sur la Theorie des Equations Integrales et ses Applications / T. Carleman // Verhandlungen des internationalen mathematiker-kongresses. – 1932. – P. 138–151.
29. Wiener, J. Generalized Solutions of Functional-Differential Equations / J. Wiener. – Singapore: World Scientic Publishing, 1999.

Фаррух Нуриддин угли Дехконов, кандидат физико-математических наук, доцент, кафедра «Математический анализ», Наманганский государственный университет (г. Наманган, Узбекистан), f.n.dehqonov@mail.ru.

*Поступила в редакцию 18 августа 2024 г.*