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MODIFICATION OF THREE-TERM CONJUGATE GRADIENT METHOD FOR SOLVING UNCONSTRAINED OPTIMIZATION AND IMAGE RESTORATION PROBLEMS

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Nonlinear conjugate gradient algorithm is highly effective for optimization due to its low storage requirements and simple structure properties. Expanding on the Barzilai and Borwein conjugate gradient method, we propose a three-term conjugate gradient method with a restart procedure for unconstrained optimization. This method ensures global convergence under standard assumptions and employs a standard Wolfe line search. To evaluate its performance, we carry out comprehensive numerical experiments for large scales to address challenges in unconstrained optimization and image restoration. The numerical results prove that the new method is more effective compared to other classical methods.

Keywords: unconstrained optimization; line search; three-term conjugate gradient method; global convergence; image restoration.

Introduction

The conjugate gradient (CG) method is one of the most prominent approaches for solving unconstrained optimization problems with high-dimensional variables. The unconstrained optimization problem of the form is the following:

$$\min_{x\in\mathbb{R}^n} f(x), \tag{1}$$

where $f:\mathbb{R}^n \to \mathbb{R}$ is continuously differentiable and its gradient is indicated by $g(x)$. The CG method for solving (1) is an iterative method that generates iteration numbers and search directions (d_k), which are given by

$$x_{k+1} = x_k + \alpha_k d_k, \quad k = 1,2,3,\dots, \tag{2}$$

$$d_k = \begin{cases} -g_k & \text{if } k=0, \\ -g_k + \beta_k d_{k-1} & \text{if } k \geq 1, \end{cases} \tag{3}$$

where x_{k+1} is the new iteration number, x_k is the current iteration number, α_k is the step size, g_k is the gradient, and β_k is the CG coefficient that differentiates the classes of CG methods.

The positive scalar α_k is commonly referred to as the step length, typically determined by inexact line search technique. One such technique is the Wolfe line search conditions, which includes two criteria:

1. The sufficient decrease condition:

$$f(x_k+\alpha_k d_k) - f(x_k) \leq \delta\alpha_k g_k^T d_k. \tag{4}$$

2. The curvature condition:

$$|g_k^T d_k| \leq -\sigma_1 g_k^T d_k. \tag{5}$$

These conditions ensure that the step length α_k yields a sufficient reduction in the objective function value while maintaining a favorable directional derivative. Here, the parameters δ and σ_1 satisfy $0 < \delta < \sigma_1 < 1$. Different choices for the conjugate parameter result in various conjugate gradient methods. These include the following classical CG-Methods, each defined by its respective conjugate parameter formula:

1. Hestenes–Stiefel (HS) method (1952) [1]:
$$\beta_k^{HS} = \frac{\langle g_k, g_k - g_{k-1}\rangle}{\langle d_{k-1}, g_k - g_{k-1}\rangle}. \tag{6}$$
2. Fletcher–Reeves (FR) method (1964) [2]:
$$\beta_k^{FR} = \frac{\langle g_k, g_k\rangle}{\langle g_{k-1}, g_{k-1}\rangle}. \tag{7}$$
3. Polak–Ribière–Polyak (PRP) method (1969) [3]:
$$\beta_k^{PRP} = \frac{\langle g_k, g_k - g_{k-1}\rangle}{\langle g_{k-1}, g_{k-1}\rangle}. \tag{8}$$
4. Conjugate-Descent (CD) method(1987) [4]:
$$\beta_k^{CD} = -\frac{\langle g_k, g_k\rangle}{\langle g_{k-1}, d_{k-1}\rangle}. \tag{9}$$
5. Liu–Storey (LS) method (1991) [5]:
$$\beta_k^{LS} = -\frac{\langle g_k, g_k - g_{k-1}\rangle}{\langle g_{k-1}, d_{k-1}\rangle}. \tag{10}$$
6. Dai–Yuan parameter (DY) method (1999) [6]:
$$\beta_k^{DY} = \frac{\langle g_k, g_k\rangle}{\langle d_{k-1}^T, g_k - g_{k-1}\rangle}. \tag{11}$$

The most important characteristics of CG methods are their global convergence and numerical performances. The methods mentioned above form two classes. The FR, CD, and DY methods have excellent global convergence, but not so good practical behavior. On the contrary, the HS, PRP, and LS methods have superior numerical performances, but they may not always converge.

As a notable extension to classical CG methods,the three-term CG methods have been the subject of considerable research focus. In Nazareth [7], the first three-term nonlinear CG-method is presented. In this paper we modified the direction which was proposed by Zhang [8], which is a modified version of the Polak–Ribiere–Polyak (PR) method, denoted as ZPR and defined by the following iteration formula:

$$d_k = -g_k + \beta_k^{PR} d_{k-1} - \frac{g_k^T d_{k-1}}{\|g_{k-1}\|^2} y_{k-1}. \tag{12}$$

1. Derivation of a New Three-Term CG Method

In this section, we introduce a new three-term conjugate gradient (CG) method designed for solving optimization problems. This method utilizes a customized vector denoted as y_k^*, which adjusts based on the Barzilai and Borwein step size α_k^{BB} [9].

We define y_{k-1}^* as a modified version of y_{k-1} given by:

$$y_{k-1}^* = \frac{1}{\alpha_k^{BB}}(1-\theta)y_{k-1}. \tag{13}$$

Here, θ is a parameter within the range (0,1) and $\alpha_k^{BB} = \dfrac{||v_{k-1}||^2}{v_{k-1}^T y_{k-1}}$.

The expression for y_{k-1}^* can be further simplified as:

$$y_{k-1}^* = (1-\theta)\frac{v_{k-1}^T y_{k-1}}{||v_{k-1}||^2}y_{k-1}. \tag{14}$$

Now, we substitute y_{k-1} with y_{k-1}^* in the third term of the equation (12): we obtain a new search direction termed as *new*:

$$d_k^{new} = \begin{cases} -g_k & \text{if } k=0, \\ -g_k + \dfrac{g_k^T y_{k-1}}{g_{k-1}^T g_{k-1}} d_{k-1} - (1-\theta)\dfrac{g_k^T d_{k-1}}{g_{k-1}^T g_{k-1}} \dfrac{v_{k-1}^T y_{k-1}}{||v_{k-1}||^2} y_{k-1} & \text{if } k>0. \end{cases} \tag{15}$$

When the condition $g_k^T d_{k-1} = 0$, the search direction outlined in equation (15) converges to the Polak–Ribière–Polyak (PRP) method.

1.1. The New-CG Algorithm

Step 1: Choose an initial point x_0 in $\mathbb{R}^n$, a tolerance $\epsilon > 0$, set $d_k = -g_k$, and initialize $k = 0$.
Step 2: If the norm of the gradient $\|g_k\|$ is less than or equal to ϵ, terminate; Otherwise, go on to the next step.
Step 3: Estimate α_k using equations (4) and(5).
Step 4: Compute the next iteration point x_k using equation (2), and update the gradient g_k and function value f_k.
Step 5: Determine the search direction d_k using equation (15).
Step 6: Check if $\|g_k\|$ is less than or equal to ϵ; if true, stop; otherwise, go to 2.
Step 7: If the termination condition is not met and certain convergence criteria are satisfied (such as ($k = n$)or ($|g_k^T g_{k-1}| \geq 0, 2\|g_k\|^2$) , return to Step 3; otherwise, Increment k by 1 and go back to Step 3 for the next iteration.

2. Convergence Analysis of the New CG Method

This section presents the convergence analysis of the new CG method. The descent condition (downhill condition) is given by the following equation:

$$g_k^T d_k < 0, \quad \forall k \geq 0, \tag{16}$$

and sufficient condition

$$g_k^T d_k \leq -c\|g_k\|^2, \quad \forall k \geq 0, \tag{17}$$

where $c > 0$.

2.1. Global Convergence of the New Method

To investigate the convergence of the new method, it is necessary to describe the key assumptions and lemma that supply the Zoutendijk condition.

Hypotheses (H):

1. The starting point $\tilde{x}$ is where the set $\mathcal{O} = \{x : x \in \mathbb{R}^n, \text{ and } \psi(x) \leq \psi(\tilde{x})\}$ is closed and bounded, $\psi(x)$ is continuously differentiable in some neighbourhood $\aleph$ of $\mathcal{O}$, and its gradient $\nabla\psi(x)$ is Lipschitz continuous, which implies that there exists a constant $\zeta > 0$ such that:

$$||\nabla\psi(\tilde{x}) - \nabla\psi(\tilde{y})|| \leq \zeta||\tilde{x} - \tilde{y}||, \quad \forall \tilde{x}, \tilde{y} \in \aleph. \tag{18}$$

2. Uniform convexity characterises the objective function $\psi(x)$, and there exists a constant $\gamma > 0$ such that:

$$(\nabla\psi(\tilde{x}) - \nabla\psi(\tilde{y}))^T(\tilde{x} - \tilde{y}) \leq \gamma||\tilde{x} - \tilde{y}||^2, \quad \forall \tilde{x}, \tilde{y} \in \aleph. \tag{19}$$

There is a constant $\eta > 0$ such that $\|\nabla\psi(\tilde{x})\| \leq \eta$, $\forall \tilde{x} \in \aleph$, under these hypotheses (H) on $\psi(x)$. Furthermore, we provide a helpful lemma that Zoutendijk basically established [10].

Lemma 1. *Given an initial value of $\breve{x}$, Hypotheses (H) must be met. Consider every method in* (2)*, assume that* (15) *defines a collection of conjugate directions, and that the step size α_k meets the typical Wolfe requirements of line search* (3) *and* (4)*. In case*

$$\sum_{k\geq 1} \frac{1}{||d_k||^2} = \infty, \tag{20}$$

then, the following

$$\lim_{k\to\infty} \inf(||g_k||) = 0 \tag{21}$$

holds.

Theorem 1. *Let the sequences of x_k and d_k be generated by algorithm 1.1, then the new direction satisfies the descent condition:*

$$d_k^T g_k \leq 0. \tag{22}$$

Proof. By mathematical induction, it will be proved, at $k = 1$, $d_1 = -g_1$, we have

$$d_1^T g_1 \leq -\|g_1\|^2 < 0.$$

Now assume that $d_{k-1}^T g_{k-1} < -\|g_{k-1}\|^2$ is true for all $k > 0$, and prove it is true at k.

To prove this, let us start with the update for the conjugate gradient direction in (15):

$$d_k = -g_k + \frac{g_k^T y_{k-1}}{g_{k-1}^T g_{k-1}} d_{k-1} - (1-\theta)\frac{g_k^T d_{k-1}}{g_{k-1}^T g_{k-1}} \frac{v_{k-1}^T y_{k-1}}{||v_{k-1}||^2} y_{k-1}. \tag{23}$$

We multiply (23) by g_k^T

$$g_k^T d_k = -g_k^T g_k + \frac{g_k^T y_{k-1}}{g_{k-1}^T g_{k-1}} g_k^T d_{k-1} - (1-\theta)\frac{g_k^T d_{k-1}}{g_{k-1}^T g_{k-1}} \frac{v_{k-1}^T y_{k-1}}{||v_{k-1}||^2} g_k^T y_{k-1}. \tag{24}$$

If the search direction is obtained through exact line search, then equation (24) satisfies the descent direction and implies that if $g_k^T d_{k-1} = 0$, then

$$g_k^T d_k = -g_k^T g_k \leq 0. \tag{25}$$

On the other hand, if d_k is not obtained through exact line search, then consider the two terms of Polak–Ribière (PR) update and achieve descent direction. Thus, we have:

$$-g_k^T g_k + \frac{g_k^T y_{k-1}}{g_{k-1}^T g_{k-1}} g_k^T d_{k-1} \leq 0.$$

Now, we need to prove the third term of (24)

$$(1-\theta)\frac{g_k^T d_{k-1}}{g_{k-1}^T g_{k-1}} \frac{v_{k-1}^T y_{k-1}}{||v_{k-1}||^2}(g_k^T y_{k-1})$$

is positive, using hypothesis 2.1to get

$$(g_k^T y_{k-1}) > g_k^T d_{k-1}.$$

So, we have

$$(1-\theta)\frac{g_k^T d_{k-1}}{g_{k-1}^T g_{k-1}} \frac{v_{k-1}^T y_{k-1}}{||v_{k-1}||^2} g_k^T y_{k-1} < (1-\theta)\frac{(g_k^T y_{k-1})^2}{g_{k-1}^T g_{k-1}} \frac{v_{k-1}^T y_{k-1}}{||v_{k-1}||^2}.$$

Since $(1-\theta) > 0$, using the Wolfe conditions (4) and (5) and lipichize condition, we have $v_{k-1}^T y_{k-1} > 0$ and $(g_k^T y_{k-1})^2 > 0$. Therefore, we can express:

$$(1-\theta)\frac{(g_k^T y_{k-1})^2}{g_{k-1}^T g_{k-1}} \frac{v_{k-1}^T y_{k-1}}{\|v_{k-1}\|^2} > 0.$$

Therefore, the new direction (15) satisfies the descent condition.

□

Theorem 2. *Let the sequences of x_k and d_k be generated by algorithm 1.1, then the new direction satisfies the sufficient descent condition:*

$$d_k^T g_k \leq -c\|g_k\|^2. \tag{26}$$

Proof. From the above theorem, we have

$$g_k^T d_k \leq -(1-\theta)\frac{g_k^T d_{k-1}}{g_{k-1}^T g_{k-1}} \frac{v_{k-1}^T y_{k-1}}{||v_{k-1}||^2} g_k^T y_{k-1}. \tag{27}$$

From the same theorem, $(1-\theta)\frac{g_k^T d_{k-1}}{g_{k-1}^T g_{k-1}} \frac{v_{k-1}^T y_{k-1}}{||v_{k-1}||^2} g_k^T y_{k-1} > 0$, so we suppose that

$$c = (1-\theta)\frac{g_k^T d_{k-1}}{g_{k-1}^T g_{k-1}} \frac{v_{k-1}^T y_{k-1}}{||v_{k-1}||^2 ||g_k||^2} g_k^T y_{k-1},$$

so (27) becomes

$$g_k^T d_k \leq -c\|g_k\|^2. \tag{28}$$

□

Theorem 3. *Suppose that assumption 2.1 holds. If x_k is obtained by algorithm 1.1, then we have*

$$\lim_{k\to\infty}(\inf \|g_k\|) = 0. \tag{29}$$

Proof. Taking the norm of both sides of(15), we have

$$\|d_k^{new}\| = \| - g_k + \frac{g_k^T y_{k-1}}{g_{k-1}^T g_{k-1}} d_{k-1} - (1-\theta)\frac{g_k^T d_{k-1}}{g_{k-1}^T g_{k-1}} \frac{v_{k-1}^T y_{k-1}}{||v_{k-1}||^2} y_{k-1}\|. \tag{30}$$

Apply preliminary Cauchy–Schwarz and triangle inequities to simplify (30), and obtain

$$\|d_k^{new}\| \le \| - g_k\| + |\frac{g_k^T y_{k-1}}{g_{k-1}^T g_{k-1}}|\|d_{k-1}\| + |(1-\theta)\frac{g_k^T d_{k-1}}{g_{k-1}^T g_{k-1}} \frac{v_{k-1}^T y_{k-1}}{||v_{k-1}||^2}|\|y_{k-1}\|. \tag{31}$$

Since $g_k^T d_{k-1} \le d_{k-1}^T y_{k-1}$, from Lipschitz condition $\|y_{k-1}\| \le L\|v_{k-1}\|$ and using $\|\nabla\psi(\tilde{x})\| \le \eta$, we have

$$\|d_k^{new}\| \le \eta + \frac{L\|g_k\|\|v_{k-1}\|}{\|g_{k-1}\|^2} + (1-\theta)(\frac{\|g_k\|\|d_{k-1}\|\|v_{k-1}\|\|y_{k-1}\|}{\|g_{k-1}\|^2\|v_{k-1}\|^2})|\|y_{k-1}\|, \tag{32}$$

$$\|d_k^{new}\| \le \eta + \frac{L\|g_k\|\|v_{k-1}\|}{\|g_{k-1}\|^2} + (1-\theta)(\frac{L^2\|g_k\|\|d_{k-1}\|\|v_{k-1}\|}{\|g_{k-1}\|^2}) \tag{33}$$

from (4) and using the form $d_{k_1} = -g_{k_1}$, we get $\|g_{k-1}\|^2 \le \frac{1}{\delta\alpha} d_{k-1}^T y_{k-1}$. Thus, rewrite (33) as

$$\|d_k^{new}\| \le \eta + \frac{\delta\alpha_k L\|g_k\|\|v_{k-1}\|}{d_{k-1}^T y_{k-1}} + (1-\theta)(\frac{\delta\alpha_k L^2\|g_k\|\|d_{k-1}\|\|v_{k-1}\|}{d_{k-1}^T y_{k-1}}) \tag{34}$$

from the (5),

$$\|d_k^{new}\| \le \eta + \frac{\delta\alpha L\eta\|v_{k-1}\|}{d_{k-1}^T y_{k-1}} + (1-\theta)(\frac{\delta\alpha L^2\eta\|d_{k-1}\|\|v_{k-1}\|}{d_{k-1}^T y_{k-1}}). \tag{35}$$

Since $d_{k-1}^T y_{k-1} \ge \sigma_1 \frac{\|v_{k-1}\|^2}{\alpha_k}$, using this form in (35) we get

$$\|d_k^{new}\| \le \eta + \frac{\delta\alpha_k^2 L\eta\|v_{k-1}\|}{\sigma_1\|v_{k-1}\|^2} + (1-\theta)(\frac{\delta\alpha_k^2 L^2\eta\|d_{k-1}\|\|v_{k-1}\|}{\sigma_1\|v_{k-1}\|^2}). \tag{36}$$

Given

$$\|v_{k-1}\| = \|x_{k+1} - x_k\|.$$

Suppose

$$\mathcal{M} = \max\{\|x_{k+1} - x_k\| : x_{k+1}, x_k \in R^n\}$$

so, (36) becomes

$$\|d_k^{new}\| \le \eta + \frac{\delta\alpha_k^2 L\eta\mathcal{M}}{\sigma_1\mathcal{M}^2} + (1-\theta)(\frac{\delta\alpha_k^3 L^2\eta\mathcal{M}^2}{\sigma_1\mathcal{M}^2}). \tag{37}$$

Suppose,

$$\begin{gathered} \gamma = \eta + \frac{\delta\alpha_k^2 L\eta}{\sigma_1\mathcal{M}} + (1-\theta)(\frac{\delta\alpha_k^3 L^2\eta}{\sigma_1}), \\ \|d_k^{new}\| \le \gamma. \end{gathered} \tag{38}$$

So, γ is a positive real number. Therefore, we can write

$$\sum_{k\ge1}\frac{1}{\|d_k^{new^*}\|^2} \ge \sum_{k\ge1}\frac{1}{\gamma^2} = \infty.$$

By using lemma, we get $\lim_{k\to\infty}\inf\|g_k\| = 0$, which completes the proof. □

3. Numerical Results and Discussion

We present here a series of numerical results concerning the new method applied to a collection of 45 functions with 450 test problems. A solver is run on a collection of p problems to obtain benchmark results. Important details, including the number of function evaluations and processing time, are noted. In order to assess and contrast the performance of the set of solvers S on a test set P, we present the idea of a performance profile in this section. We take for granted that we have n_p problems and n_s solvers. While the concepts below can be applied to other performance metrics as well, our focus is on the computation time. We define each issue P and solver S as:

$$\tau_{p,s} = \text{computation time required to solve problem P by solver S.}$$

We utilize the performance ratio to compare solver s' S performance on problem P with the best performance of all solvers on this problem.

$$r_{p,s} = \frac{\tau_{p,s}}{\min\{r_{p,s} : s \in S\}}.$$

It is assumed that for every P, S a parameter $r_M \geq r_{p,s}$ is selected, and that $r_{p,s} = r_M$ only occurs if solver S is unable to solve problem P. We shall demonstrate that the performance evaluation is unaffected by the selection of r_M. While a solver S performance on a particular problem could be interesting, we would like to get an overall evaluation of the solver's work. If we clarify

$$P_s(\tau) = \frac{1}{n_p} size\{p \in P : r_{p,s} \leq \tau\}.$$

If a performance ratio $r_{p,s}$ is within a factor $\tau \in R$ of the best achievable ratio, then the probability for solver $s \in S$is $P_s(\tau)$.The performance ratio's (accumulative) distribution function is denoted by the function p_s (Dolan and MorГк, 2002) [11]. The code for this new method is written in FORTRAN95 using the specified initial points. The results are presented in the following figures by applying the performance profiles of Dolan and More [11] to demonstrate the algorithm's efficiency. Compared to the standard HS and PRP parameter in the conjugate gradient method, the new method (NEW) demonstrates greater efficiency in terms of the number of iterations (NI) and function evaluations (NF) as shown in the Figs. 1 and 2 respectively.

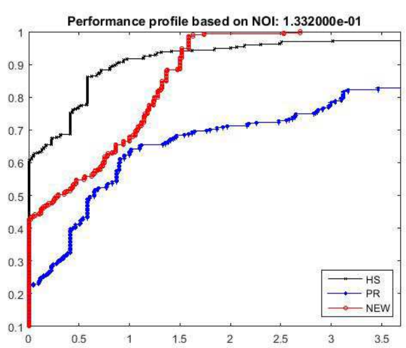

Fig. 1. Performance profile based on NI

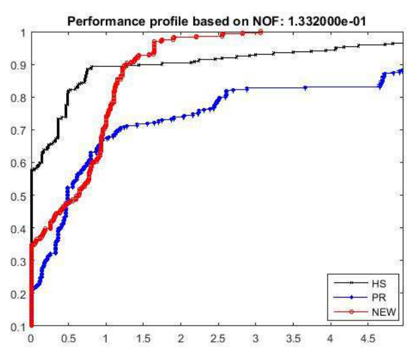

Fig. 2. Performance profile based on NF

4. Application to Image Restoration

Image restoration is an important area of study in the realm of image processing that has lately drawn interest from research groups because of its numerous applications in the security and health sectors [12]. Improving photos from a deteriorated version – usually, the noisy and blurry images – is the process of image restoration. It can be expressed mathematically [13].

The effectiveness of the suggested method on image restoration issues will be examined in this work. Using the suggested technique, the images distorted by salt-and-pepper impulse noise would be restored: LENA (512×512) and Cameraman (512×512), hill, boat, and brain. The restored image quality would be evaluated using peak signal-to-noise ratio (PSNR), which is defined as:

$$PNSR = 10.\log_{10}\left(\frac{M \times N \times 255^2}{\sum_{i,j}(X^r_{i,j} - X^*_{i,j})^2}\right), \tag{39}$$

where $X^r_{i,j}$ and $X^*_{i,j}$ represent the restored image's and the original image's pixel values, M and N are the sizes of the image [14]. Table displays the image restoration's PSNR. We discovered that the restored images produced by the suggested approach (NEW) are often higher than PRP, HS, and HZ. Fig. 3 compares the performance of the proposed method with the other three methods, which added 90% salt and pepper noise. Table displays comprehensive PSNR data for the restored image.

Fig. 3. First column: The noisy images with 90% salt-and-pepper, the original images (second column), third Column to last column: restored image by New method, HZ method,PRP method, HS method

Table

Numerical results of image restoration testing

Image	Nosie ratio %	NEW	PRP	HS	HZ
	30%	36,993069174649428	37,023036504655934	37,009657744696220	36,927257219117280
lena	50%	34,429411734127044	33,723213923376619	34,446902831435317	34,360799123081279
	90%	26,243859265298106	24,906231815588576	26,195136906369395	24,906231815588576
	30%	29,585782468936202	29,580297892936272	29,594164353587512	29,528719848633195
cameraman	50%	27,362640230151527	27,309747426689043	27,333619578052328	27,331584830182894
	90%	21,075419144261282	19,750459209731062	21,201483095676736	21,155936069934484
	30%	34,981229685095933	34,734473193084980	34,987065656749252	34,934676067228956
hill	50%	32,672393877510522	32,512447598495207	32,625474214864845	32,583040601942855
	90%	25,552443454750176	25,493125525430919	25,527291168484126	25,543367444124048
	30%	33,700053527454955	33,167401704397996	33,701642551491410	33,640779636884005
boat	50%	31,173833464752608	30,835962294902973	31,130494754286389	31,108197221963074
	90%	24,087329105510875	20,378422929846529	24,078558710540445	24,033576509731493
	30%	29,737683127069353	29,742894746913990	29,723767518709366	29,685004675274456
brain	50%	28,196987629232577	27,607722481930960	28,138532701527808	28,118419873053220
	90%	22,257949700183932	22,272347757874435	22,274004684205387	22,231223100184515

Conclusion

The proposed three-term conjugate gradient method with new direction, based on the Barzilai and Borwein technique, guarantees global convergence under standard assumptions, addressing challenges in both optimization and image restoration domains. The new formula's effectiveness was assessed by contrasting its numerical results with those of other approaches that shared the same features. Based on NI and NF, a comparison is made, and the findings indicate that the suggested method outperformed the others in terms of all parameters. We utilized the method to recover image restoration by pulse noise in order to further illustrate the effectiveness of the proposed formula. The results confirmed the power and efficiency of our method, as the new approach was able to restore images with greater precision.

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МОДИФИКАЦИЯ МЕТОДА ТРЕХСТОРОННЕГО СОПРЯЖЕННОГО ГРАДИЕНТА ДЛЯ РЕШЕНИЯ ЗАДАЧ БЕЗОГРАНИЧЕННОЙ ОПТИМИЗАЦИИ И ВОССТАНОВЛЕНИЯ ИЗОБРАЖЕНИЙ

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Нелинейный алгоритм сопряженного градиента очень эффективен для оптимизации благодаря своим низким требованиям к памяти и простым структурным свойствам. Расширяя метод сопряженных градиентов Барзилая и Борвейна, мы предлагаем метод сопряженных градиентов с тремя членами и процедурой перезапуска для безусловной оптимизации. Этот метод обеспечивает глобальную сходимость при стандартных предположениях и использует стандартный линейный поиск Вульфа. Чтобы оценить его производительность, мы проводим комплексные численные эксперименты для больших масштабов, чтобы решить проблемы безусловной оптимизации и восстановления изображений. Численные результаты доказывают, что новый метод более эффективен по сравнению с другими классическими методами.

Ключевые слова: безусловная оптимизация; линейный поиск; метод сопряженных градиентов с тремя членами; глобальная сходимость; восстановление изображений.

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