INVERSE PROBLEM OF RECOVERING FLUXES FROM INTEGRAL DATA

S.G. Pyatkov[1], *E.I. Safonov*[1], *A.A. Petrov*[1]
[1]Yugra State University, Khanty-Mansiysk, Russian Federation
E-mail: s_pyatkov@ugrasu.ru, e_safonov@ugrasu.ru, a_petrov@ugrasu.ru

In the article we consider a second order parabolic equation and well-posedness questions in Sobolev spaces of inverse problems of recovering the heat flux on the boundary with the use of a collection of integrals of a solution with weights over the domain. The flux is representable in the form of a finite segments of the series with unknown coefficients depending on time. Under certain conditions on the data, it is demonstrated that there exists a unique solution to the problem which depends on the data continuously. A solution has all generalized derivatives occurring into the equation summable to some power. The proof relies on a priori estimates and the contraction mapping principle. The method is constructive and allows to provide numerical methods of solving the problem. The numerical algorithm is based on the finite element methods and the method of finite differences. The results of numerical experiments are quite satisfactory and the procedure of constructing a solution is stable under small perturbations.

Keywords: inverse problem; boundary regime; parabolic equation; heat and mass transfer.

Introduction

We examine the parabolic system of the form

$$Mu = u_t - Lu = f(x,t), \quad (x,t) \in Q = G \times (0,T), \tag{1}$$

where $Lu = \sum_{i,j=1}^{n} \frac{\partial}{\partial x_j} a_{ij} u_{x_i} - \sum_{i=1}^{n} a_i u_{x_i} - a_0 u$, $G \subset \mathbb{R}^n$ is a bounded domain with boundary Γ, a_{ij}, a_i are matrices of dimension $h \times h$. The equation (1) is furnished with the initial-boundary conditions

$$Ru|_S = g, \ u(x,0) = u_0(x), \tag{2}$$

where $Ru = \sum_{i,j=1}^{n} a_{ij}(t,x)\nu_j \frac{\partial u}{\partial x_i} + \sigma(x,t)u$ ($\vec{\nu}$ is the unit outward normal to Γ, and the overdetermination conditions

$$\int_G \langle u(t,x), \varphi_i(x) \rangle \, dx = \psi_i(t), \ i = 1, 2, \ldots, m, \tag{3}$$

where the brackets $\langle \cdot, \cdot \rangle$ stands for the inner product in $\mathbb{R}^h$ and φ_j are linearly independent vector-functions. The function g in (2) is of the form $g = \sum_{i=1}^{r} \alpha_i(t)\Phi(t,x)$, where the functions α_i are to be determined and $\{\Phi_i\}$ are known linearly independent vector-functions. The problem is to find a solution u to the equation (1) satisfying the conditions (2), (3) and the functions $\{\alpha_i(t)\}$.

Inverse problems on determining boundary regimes arise in many of areas of mathematical physics, in particular, when modeling heat transfer processes and designing heat-protective materials, modeling the properties and thermal regimes of composite materials, modeling the thermal regimes of soils in the permafrost zone, in environmental problems (see [1–3]). The most used additional conditions are temperature measurements at some collection of points lying inside the domain or on its boundary. These problems are well-posed only in the latter case and the existence and uniqueness theorems in inverse problems of recovering the fluxes or the heat transfer coefficients with pointwise measurements are exposed in [6, 7]. The problem of recovering fluxes with arbitrary location of the measurements points is considered in [8], where the uniqueness and existence theorem was obtained under rather rigid constraints on the data. Note also the articles [9, 10], where in the model case the uniqueness theorem of a classical solution to the problem of recovering of the heat transfer coefficient $\sigma(t)$ was obtained with the use of both the pointwise condition $u(t, x_0) = \alpha(t)$ and the integral condition of the form (3). Pointwise conditions are also employed in [11] in the case of $n = 1$. In the articles [12, 13] the integral conditions of the form (3) are used as some approximation of the pointwise conditions. The problems (1) – (3) are considered in [14, 15], in the former of them some model problem was examined with $g = g(t)$,

the latter contains more general results but for a single equation. In contrast to other results, we consider inverse problem for a system of equations and our conditions for the data are a much weaker than those in [15]. Moreover, the overdetermination conditions are actually do not assume that they are valid for all components of a solution. It is possible that some of them do not occur in (3). The article is devoted to well-posedness of the problem (1) – (3). Under certain conditions it is proven that there exists a unique solution to the problem from some Sobolev space. We also describe a numerical algorithm and the results of numerical calculations.

1. Preliminaries

Let E be a Banach space. The notations for Sobolev spaces $W_p^s(G; E)$, $W_p^s(Q; E)$, etc. are standard (see [17, 18]). If $E = \mathbb{R}$ or $E = \mathbb{R}^n$ then we denote the last space simply by $W_p^s(Q)$. Definitions of Holder spaces $C^{\alpha,\beta}(\overline{Q}), C^{\alpha,\beta}(\overline{S})$ can be found, for example, in [19]. By the norm of a vector, we mean the sum of the norms of coordinates. Given an interval $J = (0, T)$, denote $W_p^{s,r}(Q) = W_p^s(J; L_p(G)) \cap L_p(J; W_p^r(G))$. Similarly, $W_p^{s,r}(S) = W_p^s(J; L_p(\Gamma)) \cap L_p(J; W_p^r(\Gamma))$. Let $(u, v) = \int\limits_G \langle u(x), v(x) \rangle \, dx$. All considered spaces and coefficients of equation (1) are assumed to be real. Next, we assume that $\Gamma \in C^2$ (see the definition in [19, Chapter 1]). Introduce the notations $Q_\gamma = (0, \gamma) \times G$ and $S_\gamma = (0, \gamma) \times \partial G$.

The space $W_p^s(0, \beta; E)$ ($s \in (0, 1)$, E is a Banach space) is endowed with the norm $\|q(t)\|_{W_p^s(0,\beta;E)} = (\|q\|_{L_p(0,\beta;E)}^p + <q>_{s,p}^p)^{1/p}$, $<q>_{s,p}^p = \int\limits_0^\beta \int\limits_0^\beta \frac{\|q(t_1)-q(t_2)\|_E^p}{|t_1-t_2|^{1+sp}} dt_1 dt_2$. If $E = \mathbb{C}$ then we obtain the usual space $W_p^s(0, \beta)$. For $s \in (0, 1)$, let $\tilde{W}_p^s(0, \beta; E) = \{q \in W_p^s(0, \beta; E) : t^{-s}q(t) \in L_p(0, \beta; E)\}$. It is a Banach space with the norm $\|q(t)\|_{\tilde{W}_p^s(0,\beta;E)}^p = \|\frac{q}{t^s}\|_{L_p(0,\beta;E)}^p + <q>_{s,p}^p$. For $s > 1/p$, all functions q from this spaces possess the property $q(\alpha) = 0$ and, for $s \neq 1/p$, the above norm and the usual norm in $\|\cdot\|_{W_p^s(0,\beta;E)}$ are equivalent for the functions $q(t)$ such that $q(0) = 0$ if $s > 1/p$ (see Lemma 1 in Sect. 3.2.6 [16]). The spaces $\tilde{W}_p^s(0, \beta; L_p(G))$, $\tilde{W}_p^{s,2s}(Q_\beta) = \tilde{W}_p^s(0, \beta; L_p(G)) \cap L_p(0, \beta; W_p^{2s}(G))$, for $s \neq 1/p$, include functions $v(t, x)$ in $W_p^s(0, \beta; L_p(G))$ and $W_p^{s,2s}(Q_\alpha)$, respectively, such that $v(0, x) = 0$ whenever $s > 1/p$. The norms $\|\cdot\|_{\tilde{W}_p^{s,2s}(Q_\beta)}$, $\|\cdot\|_{\tilde{W}_p^s(0,\beta;L_p(G))}$ are defined with the use of the above norm in $\tilde{W}_p^s(0, \beta; E)$ and we have

$$\|u\|_{\tilde{W}_p^{s,2s}(Q_\beta)}^p = \int\limits_G \int\limits_0^\beta \frac{|u(t,x)|^p}{t^{sp}} \, dt dx + \int\limits_G \int\limits_0^\beta \int\limits_0^\beta \frac{|u(t,x)-u(x,\tau)|^p}{|t-\tau|^{1+sp}} \, dt d\tau dx + \|u\|_{L_p(0,\beta;W_p^{2s}(G))}^p.$$

The spaces $\tilde{W}_p^s(\alpha, \beta; L_p(\Gamma))$ and $\tilde{W}_p^{s,2s}(S_\beta)$ are defined by analogy.

Lemma 1. *Assume that $s \in (0, 1)$, $s \neq 1/p$ and $p \in (1, \infty)$. Then*

1) the product $q \cdot v$ of functions $q \in W_p^s(0, \tau)$, $v \in W_{q_0}^{s,2s}(S_\tau) \cap L_p(\Gamma; L_\infty(0, \tau))$ ($q_0 = p$ if $s > 1/p$ and $q_0 > 1/s$ if $s < 1/p$) belongs to $W_p^{s,2s}(S_\tau)$ and we have the estimates

$$\|qv\|_{W_p^s(0,T)} \leq c_1 \|q\|_{W_p^s(0,T)}(\|v\|_{W_{q_0}^s(0,T)} + \|v\|_{L_\infty(0,T)}) \ a.a. \ on \ \Gamma,$$
$$\|qv\|_{W_p^{s,2s}(S)} \leq c_2 \|q\|_{W_p^s(0,T)}(\|v\|_{W_{q_0}^{s,2s}(S)} + \|v\|_{L_p(\Gamma;L_\infty(0,T))}); \quad (4)$$

if $q(t) \in \tilde{W}_p^s(0, \tau)$ ($\tau \in (0, T]$) and $v(t, x) \in W_{q_0}^{s,2s}(S) \cap L_\infty(Q_\tau)$ then $qv \in \tilde{W}_p^{s,2s}(S_\tau)$ and

$$\|qv\|_{\tilde{W}_p^s(0,\tau)} \leq c_3 \|q\|_{\tilde{W}_p^s(0,\tau)}(\|v\|_{W_{q_0}^s(0,\tau)} + \|v\|_{L_\infty(0,\tau)}) \leq c_4 \|q\|_{\tilde{W}_p^s(0,\tau)} \|v\|_{W_{q_0}^s(0,T)} \ a.a. \ on \ \Gamma,$$
$$\|qv\|_{\tilde{W}_p^{s,2s}(S_\tau)} \leq c_5 \|q\|_{\tilde{W}_p^s(0,\tau)}(\|v\|_{W_{q_0}^{s,2s}(S_\tau)} + \|v\|_{L_p(\Gamma;L_\infty(0,\tau))}) \leq c_6 \|q\|_{\tilde{W}_p^s(0,\tau)} \|v\|_{W_{q_0}^{s,2s}(S)}, \quad (5)$$

where the constants $c_3 - c_6$ are independent of q, v and $\tau \in (0, T]$. The claim of the lemma remains valid if we replace Γ, S with G, Q.

Proof. The proof in the case of $s > 1/p$ is exposed in Lemma 1 in [15]. Similar arguments can be used to treat the case of $s < 1/p$. So we omit the arguments.

$\square$

Fix $s_0 = 1/2 - 1/2p$ and assume that

$$a_{ij} \in C([0,T]; W_\infty^1(G)) \cap L_\infty(G; W_{q_0}^{s_0}(0,T)), \ \sigma \in W_{q_1}^{s_0,2s_0}(S) \cap L_\infty(\Gamma; W_{q_0}^{s_0}(0,T))$$
$$((n+1)/2q_1 < 1 - 1/2p), \ a_{ij}|_S \in W_{q_2}^{s_0,2s_0}(S) \ ((n+1)/2q_2 < s_0); \quad (6)$$

$$a_i \in L_\infty(G; W_{q_0}^{s_0}(0,T)) \cap L_{q_3}(Q), \ q_3 > (n+2), \ (i = 1, \ldots, n),$$
$$a_0 \in L_\infty(G; W_{q_0}^{s_0}(0,T)) \cap L_{q_4}(Q), \ q_4 > (n+2)/2, \quad (7)$$

where $i, j = 1, 2, \ldots, n$, $q_0 = p$ if $p > 3$ and $q_0 > 1/s_0$ if $p < 3$.

State the Lopatinskiy and parabolicity conditions. Consider the map $A_0(t,x,\xi) = \sum_{i,j=1}^n a_{ij}(t,x)\xi_i\xi_j$ and assume that

$$\exists \phi_A \in [0, \pi/2): \ \sigma(A_0(t,x,\xi)) \subset S_{\phi_A} = \{z \in \mathbb{C}: \ |\arg z| < \phi_A\} \quad (8)$$

for all $\xi \in \mathbb{R}^n$ with $\|\xi\| = 1$ and $(t,x) \in \overline{Q}$. Put $B_0 u = \sum_{i,j=1}^n a_{ij}\nu_j u_{x_i}$. The Lopatinskii condition is written as: given an arbitrary point $(t_0, x_0) \in S$, the boundary value problem

$$(\lambda + A_0(t_0, x_0, \xi + \nu(x_0)\partial_z))v(z) = 0, \ B_0(t_0, x_0, \xi + \nu(x_0)\partial_z)v(0) = h_j, \ z \in \mathbb{R}^+, \quad (9)$$

has a unique solution from $C(\overline{\mathbb{R}^+}; E)$ vanishing at infinity for all $\xi \in \mathbb{R}^n$, $\lambda \in \overline{S_{\pi-\phi_A}}$, and $h_j \in E$ such that $\xi \cdot \nu(x_0) = 0$ ($\nu(x_0)$ is the unit exterior normal to Γ at x_0) and $|\xi| + |\lambda| \neq 0$.

Conditions on the data:

$$f \in L_p(Q), \ u_0(x) \in W_p^{2-2/p}(G), \ p \neq 3; \quad (10)$$

$$g \in W_p^{s_0,2s_0}(S), \ g(0,x)|_\Gamma = B(0,x)u_0|_\Gamma \text{ if } p > 3; \quad (11)$$

$$\varphi_k \in W_{p'}^1(G), \Phi_k \in W_{q_0}^{s_0,2s_0}(S), \ \psi_k \in W_p^{s_0+1}(0,T), \ (f, \varphi_k) \in W_p^{s_0}(0,T), \quad (12)$$

where $q_0 = p$ if $p > 3$ and $q_0 > 1/s_0$ if $p < 3$, $k = 1, 2, \ldots, m$, and $p' = p/(p-1)$.

Theorem 1. *Let G be a bounded domain with boundary of the class C^2 and let the conditions (6) – (11) hold, where $p \neq 3$. Then there exists a unique solution u to the problem (1)-(2) such that $u \in W_p^{1,2}(Q)$. A solution satisfies the estimate*

$$\|u\|_{W_p^{1,2}(Q)} \leq C\big(\|f\|_{L_p(Q)} + \|u_0\|_{W_p^{2-2/p}(G)} + \|g\|_{W_p^{1/2-1/2p,1-1/p}(S)}\big).$$

Existence and uniqueness of solutions to the problem (1), (2) results from the known results on solvability of parabolic problems (Theorem 2.1 in [17, 21] or Theorem 5.3 in [22]).

Theorem 1 validates the following statement.

Theorem 2. *Let G be a bounded domain with boundary of the class C^2 and let the conditions (6) – (11) hold, where $f \equiv 0$ and $u_0 \equiv 0$, $p \neq 3$, $g \in \tilde{W}_p^{1/2-2/p,1-1/p}(S)$. Then a solution $u \in W_p^{1,2}(Q)$ to the problem (1), (2) meets the estimate $\|u\|_{W_p^{1,2}(Q_\gamma)} \leq c\|g\|_{\tilde{W}_p^{1/2-/2p,1-1/p}(S_\gamma)}$, where the constant c is independent of $\gamma \in (0,T)$ and g.*

Proof. The proof coincides with that in [15, Theorem 2].

$\square$

2. Main Results

In addition to the conditions of the previous section, we require that

$$|\det B(t)| \geq \delta_0 > 0 \ \forall t \in [0, T], \ \int\limits_G \langle u_0(x), \varphi_k(x) \rangle \, dx = \psi_k(0), \quad k = 1, \ldots, m, \qquad (13)$$

where $B(t)$ is the matrix with entries $b_{ij} = \int_\Gamma \varphi_i(x) \Phi_j(t, x) \, d\Gamma$;

(A) if $p > 3$ then the function $B(0, x) u_0(x)|_\Gamma$ belongs to the linear span of the functions $\Phi_1(0, x), \ldots, \Phi_m(0, x)$.

The main results of the article is the following theorem.

Theorem 3. *Assume that G is a bounded domain with boundary of the class C^2, $p \neq 3$, and the conditions (6) – (10), (12), (13), (A) hold. Then there exists a unique solution $(u, \vec{q})$ ($\vec{q} = (q_1, \ldots, q_m)$) to the problem (1) – (3) such that $u \in W_p^{1,2}(Q)$, $\vec{q} \in W_p^{s_0}(0, T)$. A solution meets the estimate*

$$\|u\|_{W_p^{1,2}(Q)} + \|\vec{q}\|_{W_p^{s_0}(0,T)} \leq$$

$$\leq C \big(\|f\|_{L_p(Q)} + \|u_0\|_{W_p^{2 - \frac{2}{p}}(G)} + \sum_{i=1}^m (\|\psi_i\|_{W_p^{1+s_0}(0,T)} + \|(f, \varphi_i)\|_{W_p^{s_0}(0,T)}) \big).$$

Proof. First, we consider the case of $p > 3$. In this case, the condition A) holds and thus there exists constants $q_i(0)$ such that $B(0, x) u_0|_\Gamma = \sum_{i=1}^m q_i(0) \Phi_i(0, x)$. In view of (13) these constants are determined uniquely. The conditions (12) and Lemma 1 imply that if $\vec{q} \in W_p^{s_0}(0, T)$ then $q_i(t) \Phi_i(t, x) \in W_p^{s_0, 2s_0}(S)$ and, respectively, $g \in W_p^{s_0, 2s_0}(S)$. In particular, $\sum_{i=1}^m q_i(0) \Phi_i(t, x) = g_0(t, x) \in W_p^{s_0, 2s_0}(S)$. Let $v \in W_p^{1,2}(Q)$ be a solution to the problem (see Theorem 1)

$$Mv = f, \quad Bv|_S = g_0(t, x), \ v|_{t=0} = u_0(x). \qquad (14)$$

Make the change of variables $u = v + w$. In this case the function $w \in W_p^{1,2}(Q)$ is a solution to the problem

$$Mw = 0, \quad Bw|_S = g - g_0 = \tilde{g}, \ w|_{t=0} = 0. \qquad (15)$$

The condition (3) can be rewritten in the form

$$\int\limits_G \langle w, \varphi_k(x) \rangle \, dx = \psi_k - \int\limits_G \langle v(t, x), \varphi_k(x) \rangle \, dx = \tilde{\psi}_k, \ k = 1, 2, \ldots, m. \qquad (16)$$

In view of (13), $\tilde{\psi}_k(0) = 0$ and, as is easily seen, $\tilde{\psi}_k(t) \in W_p^1(0, T)$. Below we shall demonstrate that $\tilde{\psi}_k(t) \in W_p^{1+s_0}(0, T)$. Multiply the equation in (15) scalarly by $\varphi_k(x)$ and integrate over G. We obtain that $(w_t, \varphi_k) = (Lw, \varphi_k)$. If $(u, \vec{q})$ is a solution to the problem (1) – (3) then integrating by parts and (15), (16) yield

$$\tilde{\psi}_k'(t) = a(w, \varphi_k) - \int\limits_\Gamma \langle \sigma w, \varphi_k \rangle \, d\Gamma + \sum_{i=1}^m \tilde{q}_i(t) \int\limits_\Gamma \langle \Phi_i, \varphi_k \rangle \, d\Gamma, \ \tilde{q}_i(t) = q_i(t) - q_i(0),$$

where $k = 1, \ldots, m$ and $a(w, \varphi_k) = -\int_G \sum_{i,j=1}^n \langle a_{ij} w_{x_i}, \varphi_{k x_j} \rangle - \langle \sum_{i=1}^n a_i w_{x_i} + a_0 w, \varphi_k \rangle \, dx$. The last equality can be rewritten in the form

$$\sum_{i=1}^m \tilde{q}_i(t) \int\limits_\Gamma \langle \Phi_i, \varphi_k \rangle \, d\Gamma = \tilde{\psi}_k'(t) - a(w, \varphi_k) + \int\limits_\Gamma \langle \sigma w, \varphi_k \rangle \, d\Gamma, \qquad (17)$$

or in the form

$$B\vec{q_a} = \vec{F}, \ \vec{F} = (F_1, \ldots, F_r)^T, \ F_k = \tilde{\psi}'_k(t) - a(w, \varphi_k) + \int_\Gamma \sigma w \varphi_k \, d\Gamma, \tag{18}$$

where $\vec{q_a} = (\tilde{q}_1, \ldots, \tilde{q}_m)^T$. The function w in (18) is a solution to the direct problem (15). The entries b_{ij} of the matrix B are such that $b_{ij} \in W_{q_0}^{s_0}(0, T)$ and, furthermore,

$$\|b_{ij}\|_{W_{q_0}^{s_0}(0,T)} \leq \|\Phi_j\|_{L_{q_0}(\Gamma; W_{q_0}^{s_0}(0,T))} \|\varphi_i\|_{L_{q'}(\Gamma)}, \ q' = q_0/(q_0 - 1), \tag{19}$$

where the parameter q_0 is that of the condition (12). Similarly,

$$\|b_{ij}\|_{L_\infty(0,T)} \leq \|\Phi_j\|_{L_\infty(0,T; L_p(\Gamma))} \|\varphi_i\|_{L_{p'}(\Gamma)} \leq c\|\Phi_j\|_{L_p(\Gamma; L_\infty(0,T))}, \ p' = p/(p - 1), \tag{20}$$

By Lemma 1, the products of any number of the entries b_{ij} belongs to $W_{q_0}^{s_0}(0, T)$ and thus the entries of the matrix B^{-1} also possess this property. The condition (13) yields

$$\vec{q_a} = B^{-1}\vec{F} = R(\vec{q_a}) = \vec{g} + R_0(\vec{q_a}), \tag{21}$$

where $\vec{g} = B^{-1}\vec{\Psi}$, $\vec{\Psi} = (\Psi_1, \Psi_2, \ldots, \Psi_r)^T$, $\Psi_k(t) = \tilde{\psi}'_k(t)$. Next, we prove that this equation is solvable, at least on some small time interval. Consider the segment $[0, \delta] \subset [0, T]$. Estimate $\|R_0(\vec{q_a})\|_{\tilde{W}_p^{s_0}(0,\delta)}$. By Lemma 1, we infer

$$\|R_0(\vec{q_a})\|_{\tilde{W}_p^{s_0}(0,\delta)} \leq c \sum_{k=1}^m (\|a(w, \varphi_k)\|_{\tilde{W}_p^{s_0}(0,\delta)} + \|\int_\Gamma \langle \sigma w, \varphi_k \rangle \, d\Gamma\|_{\tilde{W}_p^{s_0}(0,\delta)}). \tag{22}$$

By definition,

$$a(w, \varphi_k) = -\int_G \sum_{i,j=1}^n \langle a_{ij} w_{x_i}, \varphi_{k x_j} \rangle + \langle (\sum_{i=1}^n a_i w_{x_i} + a_0 w), \varphi_k \rangle \, dx.$$

The Minkowki and Hölder inequalities and Lemma 1 yield

$$J = \|\int_G \langle a_{ij} w_{x_i}, \varphi_{k x_j} \rangle \, dx\|_{\tilde{W}_p^{s_0}(0,\delta)} \leq c_1 \left(\int_G \|\nabla w\|_{\tilde{W}_p^{s_0}(0,\delta)}^p \, dx \right)^{\frac{1}{p}} \leq c_1 \|w\|_{\tilde{W}_p^{s_0}(0,\delta; W_p^1(G))}. \tag{23}$$

If $\tilde{w}$ is the zero extension of w to the interval $(-\infty, \delta)$ then it is easy to check that the norms $\|w\|_{\tilde{W}_p^{s_0}(0,\delta; W_p^1(G))}$ and $\|\tilde{w}\|_{W_p^{s_0}(-\infty,\delta; W_p^1(G))}$ are equivalent with constants independent of δ. The definition of the norm and [18, Theorems 4.1.3, 4.5.5 of Ch. 7] yield

$$\|w\|_{\tilde{W}_p^{s_0}(0,\delta; W_p^1(G))} \leq c_2 \|\tilde{w}\|_{W_p^{s_0}(-\infty,\delta; W_p^1(G))} \leq c_2 \delta^{\gamma_0} \|\tilde{w}\|_{W_p^{s_0+\gamma_0}(-\infty,\delta; W_p^1(G))} \leq$$
$$c_3 \delta^{\gamma_0} \|\tilde{w}\|_{H_p^{s_0+\gamma_0+\varepsilon_0}(-\infty,\delta; W_p^1(G))} \leq c_4 \delta^{\gamma_0} \|\tilde{w}\|_{W_p^{1,2}(Q_\infty)} \leq c_4 \delta^{\gamma_0} \|w\|_{W_p^{1,2}(Q_\delta)}, \tag{24}$$

where $0 < \gamma_0 < 1/2 - s_0 - \varepsilon_0$, $\varepsilon_0 > 0$, and $Q_\infty = (-\infty, \delta) \times G$. The relation (24) validates the estimate

$$\|\int_G \langle a_{ij} w_{x_j}, \varphi_{k x_i} \rangle dx\|_{\tilde{W}_p^{s_0}(0,\delta)} \leq c_6 \delta^{\gamma_0} \|w\|_{W_p^{1,2}(Q^\delta)}, \tag{25}$$

where the constant c_6 is independent of δ. The summands $I_1 = \int_G \langle a_i w_{x_i}, \varphi_k \rangle \, dx$ and $I_2 = \int_G \langle a_0 w, \varphi_k \rangle \, dx$ in $a(w, \varphi_k)$ are estimated by analogy and the estimates are simpler. So we omit them. Proceed with the last summand $J_1 = \|\int_\Gamma \langle \sigma w, \varphi_k \rangle \, d\Gamma\|_{\tilde{W}_p^{s_0}(0,\delta)}$. We have (see (24)) that

$$J_1 \leq c_1 \|w\|_{\tilde{W}_p^{s_0}(0,\delta; L_p(\Gamma))} \leq c_2 \|w\|_{\tilde{W}_p^{s_0}(0,\delta; W_p^1(G))} \leq c_3 \delta^{\gamma_0} \|w\|_{W_p^{1,2}(Q_\delta)}. \tag{26}$$

36

Bulletin of the South Ural State University. Ser. Mathematical Modelling, Programming & Computer Software (Bulletin SUSU MMCS), 2025, vol. 18, no. 4, pp. 32–44

Here we have used the embedding $W_p^1(G) \subset L_p(\Gamma)$ and the Hölder inequality. The estimates (25), (26) and the corresponding estimates for I_1, I_2 imply that

$$\|R_0(\vec{q_a})\|_{\tilde{W}_p^{s_0}(0,\delta)} \leq c_4 \delta^{\gamma_0} \|w\|_{\tilde{W}_p^{1,2}(Q_\delta)}, \tag{27}$$

where the constant c_4 is independent of δ. Theorem 2 yields

$$\|w\|_{W_p^{1,2}(Q_\delta)} \leq c\|\tilde{g}\|_{\tilde{W}_p^{s_0,2s_0}(S_\delta)}. \tag{28}$$

By Lemma 1, $\|\tilde{g}\|_{\tilde{W}_p^{s_0,2s_0}(S_\delta)} \leq c_1 \sum_{i=1}^m \|\tilde{q}_i\|_{\tilde{W}_p^{s_0}(0,\delta)}$. Here the constant c_1 depends on the norms $\|\Phi_i\|_{W_{q_0}^{s_0,2s_0}(S)}$. In this case it follows from (27), (28) that

$$\|R_0(\vec{q_a})\|_{\tilde{W}_p^{s_0}(0,\delta)} \leq c_5 \delta^{\gamma_0} \sum_{i=1}^m \|\tilde{q}_i\|_{\tilde{W}_p^{s_0}(0,\delta)} = c_5 \delta^{\gamma_0} \|\vec{q_a}\|_{\tilde{W}_p^{s_0}(0,\delta)}, \tag{29}$$

where the constant c_5 is independent of δ and $\vec{q_a}$. Thereby, if $\delta_0^{\gamma_0} c_5 < 1$ then the operator R_0 is a contraction and, by the fixed point theorem, the equation (21) has a unique solution from the space $W_p^{s_0}(0, \delta_0)$ under the condition that $\tilde{\psi}_k' \in \tilde{W}_p^{s_0}(0, T)$. In view of the conditions of the theorem $\psi_k' \in W_p^{s_0}(0, T)$. Demonstrate that $\psi_{0k} = \int_G v(t, x)\varphi_k(x)\, dx \in W_p^{1+s_0}(0, T)$, i. e., $\int_G v_t(t, x)\varphi_k(x)\, dx \in W_p^{s_0}(0, T)$. Multiply the equation in (14) by φ_k and integrate the result over G. We obtain that

$$\psi_{0k}'(t) = a(v, \varphi_k) - \int_\Gamma \sigma v\varphi_k\, d\Gamma + \sum_{i=1}^m \tilde{q}_i(0) \int_\Gamma \Phi_i \varphi_k\, d\Gamma + (f, \varphi_k), \ k = 1, \dots, m. \tag{30}$$

Repeating the arguments used in the proof of the estimate for $\|R_0\vec{q_a}\|_{\tilde{W}_p^{s_0}(0,\delta)}$, with $\delta = T$, we can show that the right-hand side in the above equality belongs to $W_p^{s_0}(0, T)$ and, thus, $\psi_{0k} \in W_p^{1+s_0}(0, T)$. We have proven that the equation (21) has a unique solution (in correspondiong ball) on the segment $[0, \delta_0]$. Next, we can define a solution $w \in W_p^{1,2}(Q_{\delta_0})$ to the problem (15). Demonstrate that the conditions (16) hold. Multiply the equation in (15) by φ_k and integrate over G. Integrating by parts and the estimates (14), (15) imply that

$$\int_G \langle w_t, \varphi_k \rangle\, dx = a(w, \varphi_k) - \int_\Gamma \langle \sigma w, \varphi_k \rangle\, d\Gamma + \sum_{i=1}^m \tilde{q}_i(t) \int_\Gamma \langle \Phi_i, \varphi_k \rangle\, d\Gamma, \ k = 1, \dots, m.$$

The vector-function $\vec{q_a}$ meets the system (17), summing its kth equation with the equality obtained, we derive that $\int_G \langle w_t, \varphi_k \rangle\, dx = \tilde{\psi}_k', \ k = 1, \dots, m$. Integrating this equality with respect to t and using the initial condition, we justify (16) on $[0, \delta_0]$. Next, we show that a solution is extendible to the whole segment $[0, T]$. We have defined the vector-function $\vec{q_a}$ only on $[0, \delta_0]$. Extend the vector-function $\vec{q_a}$ by zero for $t < 0$ and assign $\vec{q_b} = \begin{cases} \vec{q_a}(t), \ t \in (0, \delta_0) \\ \vec{q_a}(2\delta_0 - t), \ t \in [\delta_0, T] \end{cases}$ Denote by $q_1^b, \dots, q_r^b$ the coordinates of the vector $\vec{q_b}$. The new vector-function $\vec{q_b}$ belongs to $W_p^{s_0,2s_0}(S)$. Make the change of variables $\vec{q}^1 = \vec{q_a} - \vec{q_b}$. This vector-function with the coordinates q_i^1 satisfies the system of equations

$$\sum_{i=1}^m q_i^1(t)b_{ki} = \tilde{\psi}_k'(t) - a(w, \varphi_k) + \int_\Gamma \langle \sigma w, \varphi_k \rangle\, d\Gamma - \sum_{i=1}^m q_i^b(t)b_{ki}.$$

In view of the definition, the vector $\vec{q}^1$ vanishes on $[0, \delta_0]$. Let w_0 be a solution to the problem $Lw_0 = 0$, $Bw_0|_S = \sum_{i=1}^m q_i^b \Phi_i$, $w_0|_{t=0} = 0$. In this case the function $w_1 = w - w_0$ is a solution to the problem

$$Lw_1 = 0, \quad Bw_1|_S = \sum_{i=1}^m q_i^1 \Phi_i, \ w_1|_{t=0} = 0. \tag{31}$$

By Theorem 1, $w_1 = 0$ for $t \in [0, \delta_0]$. Thus, the extension problem for the function $\vec{q_a}$ is reduced to determining a solution to the system

$$\sum_{i=1}^m q_i^1(t) b_{ki} = \psi'_{1k}(t) - a(w_1, \varphi_k) + \int_\Gamma \sigma w_1 \varphi_k \, d\Gamma, \tag{32}$$

where $\psi'_{1k} = \tilde{\psi}'_k(t) - a(w_0, \varphi_k) + \int_\Gamma \langle \sigma w_0, \varphi_k \rangle \, d\Gamma - \sum_{i=1}^m q_i^b(t) b_{ki}$, and the function w_1 is a solution to the problem (31). This soluton vanishes for $t \leq \delta_0$. We have obtained the same system as before, but the Cauchy data are given at $t = \delta$ and the right-hand side of the system (more exactly the vector $\vec{g}$) differs from the previous one. Next, we can repeat the arguments of the previous proof on the segment $[\delta_0, 2\delta_0]$. They are the same and without loss of generality we can assume that the constants arising in the proof of the estimate for the norm of the operator R_0 are the same. Thus the system (32) is solvable on the segment $[\delta_0, 2\delta_0]$. Repeating the arguments on $[2\delta_0, 3\delta_0]$, etc. we construct a solution on the whole segment $[0, T]$. The estimate from the statement of the theorem was actually obtained in the proof of the theorem. Consider the case of $p < 3$. Take $g_0 = 0$ in (14) and $\tilde{g} = g$ in (15). Next, we repeat the arguments without any distinctions.

$\square$

3. Numerical Algorithm

The numerical algorithm employs the idea of the proof of Theorem 3. The equation under consideration is as follows:

$$Mu = u_t - Lu = u_t - div(c(x,t)\nabla u) + b(x,t)\nabla u + a(x,t)u = f,$$
$$b(x,t) = (b_1(x,t), \dots, b_n(x,t))^T, \nabla u = (\tfrac{\partial u}{\partial x_1}, \dots, \tfrac{\partial u}{\partial x_n})^T, n = 2, 3, \tag{33}$$

where $(t, x) \in Q = (0, T) \times G$ and c is a diagonal matrix. Below we consider the case of $n = 2$ and assume that $G = (0, X) \times (0, Z)$, $(X, Z > 0)$. As before, $\Gamma = \partial G$, let $\Gamma_0 = \{(x_1, 0) : x_1 \in (0, X)\}$, $S = (0, T) \times \Gamma$, $S_0 = (0, T) \times \Gamma_0$. The additional measurements and the initial and boundary conditions are of the form

$$\int_G u(y,t) \Psi_i(x) \, dx = \psi_i(t), \ i = 1, 2, \dots, r, \tag{34}$$

$$u|_{t=0} = u_0(x), \ c_2 u_{x_2}|_{S_0} = g(t, x), \ u|_{S \setminus S_0} = 0, \ \text{or} \ \frac{\partial u}{\partial N} + \sigma u|_{S \setminus S_0} = 0. \tag{35}$$

We employ the method of finite elements (FEM). The unknown function g is sought in the form $g = \sum_{j=1}^r \alpha_j(t) \Phi_j(x_1)$, where the functions α_i are unknowns. We assume that

$$B_0 = \det\{b_{ij}\}, \ b_{ij} = \int_0^X \Phi_j(x_1) \Psi_i(x_1, 0) \, dx_1, \ det B_0 \neq 0.$$

Given a triangulation of G and the corresponding basis $\{\varphi_i\}_{i=1}^N$ which is defined in accord with the choice of the boundary conditions in (35). Denote the nodes of the mesh by $\{b_i\}$. An approximate solution is sought in the form $v = \sum_{i=1}^N C_i(t)\varphi_i$. The functions C_i are determined from the system

$$M\vec{C}_t + K(t)\vec{C} = -\vec{F} + \vec{f}, \ \vec{C} = (C_1, C_2, \dots, C_N)^T, \ i = 0, 1, 2, \dots, N, \tag{36}$$

where

$$\vec{F} = (\int\limits_0^X g(t,x_1)\varphi_1(x_1,0)\,dx_1, \ldots, \int\limits_0^X g(t,x_1)\varphi_N(x_1,0)\,dx_1)^T,$$

$$\vec{f} = (f_1, \ldots, f_r)^T, \ \ f_i = \int\limits_G f(t,x)\varphi_i(x)\,dx.$$

The matrix M and K have the entries $m_{ij} = (\varphi_i, \varphi_j) = \int\limits_G \varphi_i(x)\varphi_j(x)\,dx,$

$$K_{jk} = (c_1(t,x)\varphi_{kx_1}, \varphi_{jx_1}) + (c_2(t,x)\varphi_{kx_2}, \varphi_{jx_2}) + (b(t,x)\nabla\varphi_k, \varphi_j) + (a(t,x)\varphi_k, \varphi_j). \quad (37)$$

If we consider the Robin boundary condition on $S \setminus S_0$ then to the above summands we need to add the quantity $\int\limits_{S\setminus S_0} \sigma\varphi_k\varphi_j\,d\Gamma$. We have that $\vec{C}(0) = \vec{C}_0 = (u_0(b_1), \ldots, u_0(b_N))$.

A solution to the system (36) is sought by the method of finite differences. Let $\tau = T/M$ is time step. We replace the equation (36) with the system

$$M\frac{\vec{C}_{i+1} - \vec{C}_i}{\tau} + K_{i+1}\vec{C}_{i+1} = -\vec{F}_{i+1} + \vec{f}_{i+1}, \ \ \vec{C}_i = (C_i^1, \ldots, C_i^N)^T, \ i = 0, 1, \ldots, M-1, \quad (38)$$

where $C_i^k \approx C_k(\tau i)$, $\vec{F}_i \approx \vec{F}(\tau i)$, $\vec{f}_i = \vec{f}(\tau i)$, $K_i = K(\tau i)$.

Introduce the following notations: $\vec{\psi}(t) = (\psi_1(t), \psi_2(t), \ldots, \psi_r(t))^T$, $\vec{\psi}_i = \vec{\psi}(\tau i)$, $\psi_i^k = \psi_k(\tau i)$, $\vec{\alpha}_i = (\alpha_i^1, \ldots, \alpha_i^r)^T$, $\vec{\alpha}_i \approx \vec{\alpha}(\tau i)$, $\alpha_i^k \approx \alpha_k(i\tau)$, $\vec{\Psi}_k = (\psi_k(b_1), \psi_k(b_2), \ldots, \psi_k(b_N))^T$.

The analog of the overdetermination conditions (34) is the equalities $\langle M\vec{C}_i, \vec{\Psi}_k \rangle = \psi_i^k$, where the brackets stand for the inner product in $\mathbb{R}^N$. The initial data are as follows: $C_0^k = u_0(b_k)$ for $k = 1, \ldots, N$.

The vector $\vec{F}_i$ can be written in the form $A\vec{\alpha}_i$, where the entries of A are as follows: $a^{ij} = \int\limits_0^X \Phi_j(x_1)\varphi_i(x_1,0)\,dx_1$ or $\vec{F}_i = \sum_{j=1}^r \vec{e}_j\alpha_i^j$, with $e_j^k = \int\limits_0^X \Phi_j(x_1)\varphi_k(x_1,0)\,dx_1$.

The system (38) can be written in the form

$$R_{i+1}\vec{C}_{i+1} = -\tau\vec{F}_{i+1} + \tau\vec{f}_{i+1} + M\vec{C}_i, \ i = 0, 1, 2, \ldots, M-1, \quad (39)$$

where $R_{i+1} = M + \tau K_{i+1}$. Inverting the matrix R_{i+1} and applying M, we infer

$$M\vec{C}_{i+1} = -\tau M(R_{i+1})^{-1}\vec{F}_{i+1} + M(R_{i+1})^{-1}(\tau\vec{f}_{i+1} + M\vec{C}_i). \ i = 0, 1, 2, \ldots, M-1, \quad (40)$$

Multiply the system (40) scalarly by $\vec{\Psi}_k$. We obtain that

$$\psi_{i+1}^k = -\tau\langle M(R_{i+1})^{-1}\vec{F}_{i+1}, \vec{\Psi}_k \rangle + \langle M(R_{i+1})^{-1}(\tau\vec{f}_{i+1} + M\vec{C}_i), \vec{\Psi}_k \rangle. \quad (41)$$

If e_j are the columns of A, $\vec{F}_{i+1} = \sum_{l=1}^r e_l\alpha_{i+1}^l$, then this system can be written in a matrix form. Let the matrix B_{i+1} has the entries $b_{kl} = \langle M(R_{i+1})^{-1}e_l, \vec{\Psi}_k \rangle$. In this case the system (41) is rewritten as follows:

$$\vec{\psi}_{i+1} + \vec{H}_{i+1} = -\tau B_{i+1}\vec{\alpha}_{i+1}, \ H_{i+1}^k = -\langle M(R_{i+1})^{-1}(\tau\vec{f}_{i+1} + M\vec{C}_i), \vec{\Psi}_k \rangle. \quad (42)$$

Decribe the sequence of the calculations. First, we determine the initial data $C_0^k = u_0(b_k)$. Assume that we have calculated $\vec{\alpha}_i$, $\vec{C}_i$. The vector $\vec{\alpha}_{i+1}$ is defined from (42). Inserting it in (38), we can find $\vec{C}_{i+1}$.

4. Programm Realisation

In this section we present the results of numerical experiments. To determine the accuracy of calculations, we take given functions c, $\vec{b}$, a, f and the function g depending on the known functions Φ_i and α_i. By solving the direct problem (33), (35) we can find a solution u of this problem and thereby find the values (34). Next, using the data of the direct problem and overdetermination conditions, we can solve the inverse problem (33) – (35), and find the solution u and the functions α_i. By comparing the original functions α_i, u and the results of calculation, one can evaluate the convergence of the algorithm.

The characteristics of the computer are as follows: Processor: Intel(R) Xeon(R) CPU E5-2678 v3 @ 2.50GHz (2 two processors); RAM: 64.0 GB; OS: Windows 10 Pro x64.

We will use a rectangle with vertices $(0; 0)$, $(2, 0)$, $(2; 1)$, and $(0, 1)$ as the domain G. Let's create a mesh using Delaunay triangulation. We employ four grids $Z_1 = 103$, $Z_2 = 379$, $Z_3 = 1453$ and $Z_4 = 5689$. Having obtained the coordinates of the grid nodes, it is possible to calculate arrays for known input functions (using symbolic computing software packages). Also we define the arrays of the nodes on sides of the rectangle where $y = 0$ and $x \neq 0$, $x \neq 2$. Note that we use the homogeneous Dirichlet condition on all side except for the lower side of the rectangle.

Let the time step be equal to $\tau = T/m$, with $m = 100$. The next step is constructing the functions $\vec{\psi}_i$: we take the solution of the direct problem $\vec{C}_i^{dp}$ and define $\psi_i^k = \langle M \cdot \vec{C}_i^{dp}, \vec{\Psi}_k \rangle$ $(k = 1, 2, \ldots, r; i = 1, 2, \ldots, m)$ as described in (34). We can also add random noise to a solution to the direct problem u_d. Noise is represented as a random variable with the range $n_s \in (-\delta; \delta)$, where δ is the percentage deviation. The resulting function is of the form $\psi_i(j\tau) = \langle M \cdot \vec{C}_i^{dp} \cdot (1 + n_s(j\tau)/100), \vec{\Psi}_i \rangle$. Introduce some quantities: the equality $\varepsilon_\alpha = \max_i(\max_j |\alpha^j(i\tau) - \alpha_i^j|)$ defines the calculation error for the functions α_i (the numbers α_i^j are the result of calculations, $\alpha_i^j \approx \alpha^j(\tau i)$, $j = 1, 2, \ldots, r$); the error of calculations of a solution $C_{i,j}$ to the inverse problem is defined as $\varepsilon_u = \max_{i,j} |C_{i,j}^{dp} - C_{i,j}|$, where $i = 1, 2, \ldots, N$ and $j = 1, 2, \ldots, m$; T_τ is the execution time of the algorithm in seconds.

For the first group of the data, we take coefficients of the equation: $a = 1/(t + 1)$, $b_1 = 0$, $b_2 = 0$, $c_{11} = x + 2$, $c_{22} = y + 2$, $f = 1$; additional measurements $\Psi_1 = 1$, $\Psi_2 = |x|$ and the desired functions $\alpha_1 = t + 1$, $\alpha_2 = (t + 1)^2 - 1$. If we take $\delta = 0$ then the graphics of the initial and the constructed functions actually coincide. So let's present the results in triplet form $(\varepsilon_\alpha; \varepsilon_u; T_\tau)$ for grids Z_l (Table 1).

Table 1

<p align="center">Results for first group</p>

grid	ε_α	ε_u	T_τ
Z_1	$9,39e^{-14}$	$4,21e^{-15}$	$10,7$
Z_2	$1,63e^{-13}$	$6,88e^{-15}$	$9,7$
Z_3	$9,1e^{-14}$	$4,88e^{-15}$	$51,7$
Z_4	$7e^{-14}$	$3,1e^{-15}$	$1048,3$

Based on the results of the experiments of the first group, it is difficult to estimate the dependence of the increase in the accuracy of calculations with an increase in the number of grid nodes. For the second group of data, we take data from the first group, add noise $\delta = 0.05$, add additional measurement $\Psi_3 = x^2$ and the function $\alpha_3 = (t - 2)^2 + 2$. As for the previous group, we will present the results in the form of triplets, (Table 2).

Table 2

<p align="center">Results for second group</p>

grid	ε_α	ε_u	T_τ
Z_1	$1,66$	$0,09$	$3,88$
Z_2	$1,34$	$0,05$	$9,45$
Z_3	$0,7$	$0,03$	$49,3$
Z_4	$0,39$	$0,01$	$1061,2$

The results shown in Fig. 1. For brevity, we only display graphics and tables with the results of calculations α_i.
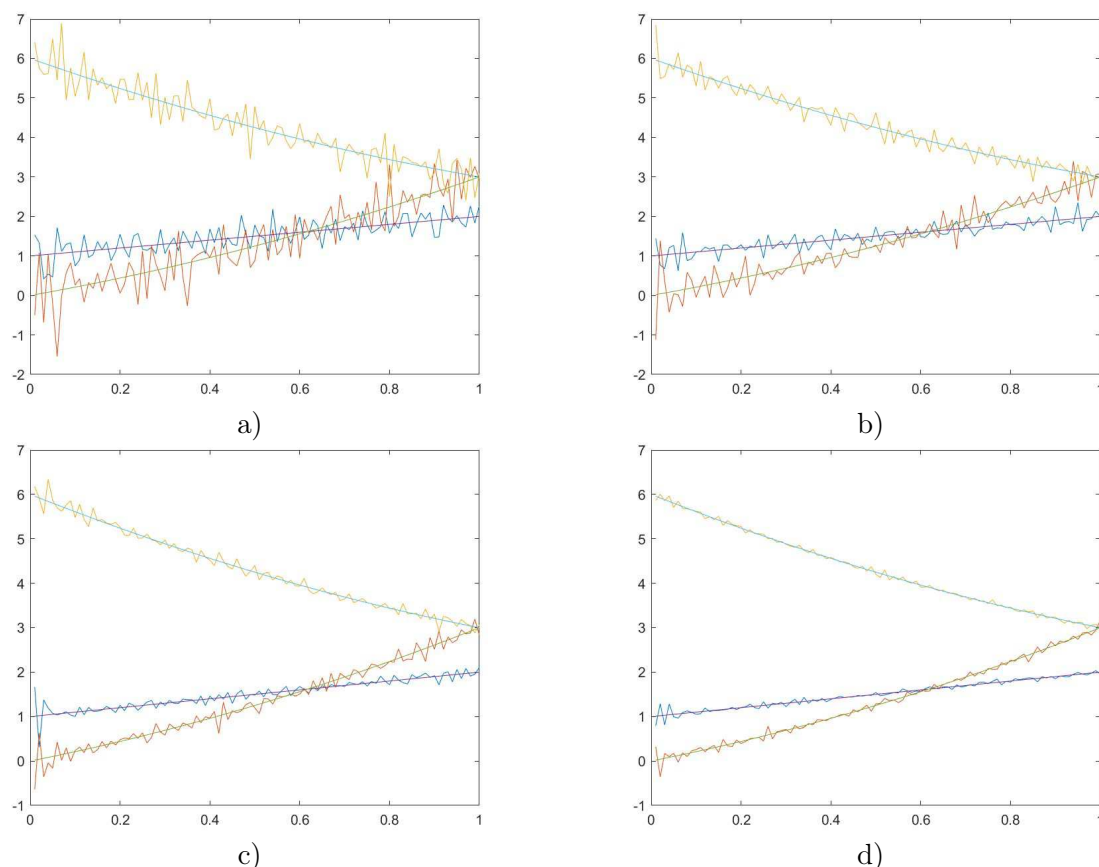
Fig. 1. The results for second group of data: a) Z_1; b) Z_2; c) Z_3; d) Z_4

As we can see 5-percent random noise increases the calculation error quite significantly, but despite this, the calculation results repeat the trends of the desired functions. It is now clear that increasing the number of nodes has a positive effect on accuracy, but a negative effect on the computation time. Based on this, for further experiments we will use only grid Z_3.

For the third group of data, we take coefficients of the equation: $a = (x+1)(y+1)/(1+t)$, $b_1 = (t+1)(x+1)^2$, $b_2 = (t+1)(y+1)^2$, $c_{11} = (x+1)/(t+1)$, $c_{22} = (y+1)/(t+1)$, $f = 0$; the desired functions $\alpha_1 = -t + 2$, $\alpha_2 = 3(t-0,6)^2 + 1$, $\alpha_3 = 3(t-0,5)^3 + 1,5$; additional measurements and random noise are the same as in the second group.

The graphs are shown in Fig. 2 and the result is the triplet: $(0,66; 0,05; 42,9)$.
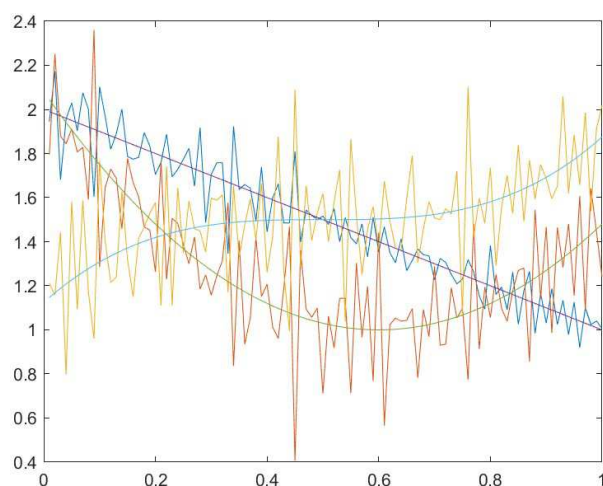

Fig. 2. The results for third group of data

Conclusion

Using theoretical results on well-posedness of the problem, we construct a numerical algorithm for recovering the flux on the boundary with the use of integral observations of the concentration. It is based on the conventional methods (in our case FEM and difference schemes). The results of numerical experiments are presented. The obtained results reveal the accuracy, efficiency, and robustness of the proposed algorithm. It is stable under random perturbations of the data.

References

1. Alifanov O.M., Artyukhin E.A., Nenarokomov A.V. *Inverse Problems in the Study of Complex Heat Transfer.* Moscow, Janus-K, 2009.
2. Permyakov P.P., Afanasyeva T.A., Varlamov S.P., Skryabin P.N. Recovering of Boundary Conditions for Heat Exchange Modeling on the Ground Surface. *Arctic XXI Century. Humanitarian Sciences*, 2019, vol. 17, no. 1, pp. 27–35. DOI: 10.25587/2309-4412-2019-1-27-35
3. Permjakov P.P. *Identifikation of Parameters of a Heat and Moisture Transfer Model in Frozen Soils.* Novosibirsk, Nauka, 1989.
4. Sabrekov A.F., Glagolev M.V., Terentyeva I.E. Determination of the Specific Flux of Methane from Oil Using Inverse Modeling Based on Conjugate Equations. *Reports of the International Conference Mathematical Biology and Bioinformatics*, 2018, vol. 7, no. e94. DOI: 10.17537/icmbb18.85
5. Belolipeckij V.M., Belolipeckij P.V. Estimate of the Carbon Dioxide Flux between the Atmosphere and the Terrestrial Ecosystem with Tower Measurements of Vertical Concentration Distribution of CO_2. *Bulletin of the Novosibirsk State University. Series Information Technologies*, 2011, vol. 9, no. 1, pp. 75–81.
6. Pyatkov S.G., Baranchuk V.A. Determination of the Heat Transfer Coefficient in Mathematical Models of Heat and Mass Transfer. *Mathematical Notes*, 2023, vol. 113, no. 1, pp. 93–108. DOI: 10.4213/mzm13573
7. Pyatkov S.G., Baranchuk V.A. Identification of a Boundary Condition in Heat and Mass Transfer Problems. *Chelyabinsk Physical and Mathematical Journal*, 2022, vol. 7, no. 2, pp. 234–253. DOI: 10.47475/2500-0101-2022-17206
8. Pjatkov S.G., Potapkov A.A. The Well-Posed Identification of the Interface Heat Transfer Coefficient Using an Inverse Heat Conduction Model. *Mathematics*, 2023, vol. 11, no. 23, pp. 1–14. DOI: 10.3390/math11234739
9. Kostin A.B., Prilepko A.I. On Some Problems of the Reconstruction of a Boundary Condition for a Parabolic Equation. II. *Differential Equations*, 1996, vol. 32, no. 11, pp. 1515–1525.
10. Kostin A.B., Prilepko A.I. On some Problem of the Reconstruction of a Boundary Condition for a Parabolic Equation. I. *Differential Equations*, 1996, vol. 32, no. 1, pp. 113–122.
11. Pilant M., Rundell W. An Iteration Method for the Determination of an Unknown Boundary Condition in a Parabolic Initial-Boundary Value Problem. *Proceedings of the Edinburgh Mathematical Society*, 1989, vol. 32, issue 1, pp. 59–71. DOI: 10.33048/smzh.2024.65.410
12. Hao Dinh Nho, Thanh Phan Xuan, Lesnik D. Determination of the Heat Transfer Coefficients in Transient Heat Conduction. *Inverse Problems*, 2013, vol, 29, no. 9, P. 095020. DOI: 10.1088/0266-5611/29/9/095020
13. Hao Dinh Nho, Huong Bui Viet, Thanh Phan Xuan, Lesnik D. Identification of Nonlinear Heat Transfer Laws from Boundary Observations. *Applicable Analysis*, 2014, vol. 94, no. 9, pp. 1784–1799. DOI: 10.1080/00036811.2014.948425
14. Kozhanov A.I. Linear Inverse Problems for Some Classes of Nonlinear Nonstationary Equations. *Siberian Electronic Reports*, 2015, vol. 12, pp. 264–275.
15. Verzhbitskii M.A., Pyatkov S.G. On some Inverse Problems of Recovering the Bounary Regimes. *Mathematical notes of NEFU*, 2016, vol. 23, no. 2, pp. 3–18.
16. Triebel H. Interpolation Theory. Function Spaces. Differential Operators. *Veb Deutscher Verlag des Wissenschaften*, Berlin, 1978.
17. Denk R., Hieber M., Pruess J. Optimal $L_p - L_q$-Estimates for Parabolic Boundary Value Problems with Inhomogeneous Data. *Mathematische zeitschrift*, 2007, vol. 257, no. 1, pp. 193–224. DOI: 10.1007/s00209-007-0120-9
18. Amann H. Compact Embeddings of Vector-Valued Sobolev and Besov Spaces. *Glasnik matematicki*, 2000, vol. 35, no. 1, pp. 161–177.
19. Ladyzhenskaya O.A., Solonnikov V.A., Ural'tseva N.N. Linear and Quasi-Linear Equations of Parabolic Type. *Translations of Mathematical Monographs*, 1968, vol. 23. DOI: 10.1090/mmono/023
20. Behzadan A. Multiplication in Sobolev Spaces, Revisited. *Arkiv for Matematik*, 2021, vol. 59, pp. 275–306. DOI: 10.4310/ARKIV.2021.v59.n2.a2

21. Meyries M., Schnaubert R. Maximal Regularity with Temporal Weights for Parabolic Problems with Inhomogeneous Boundary Conditions. *Mathematische Nachrichten*, 2012, vol. 285, no. 8–9, pp. 1032–1051. DOI: 10.1002/mana.201100057

22. Hummel F., Lindemulder N. Elliptic and Parabolic Boundary Value Problems in Weighted Function Spaces. *Potential Analysis*, 2022, vol. 57, pp. 601–669. DOI: 10.1007/s11118-021-09929-w

23. Grisvard P. Equations Differentielles Abstraites, *Annales scientifiques de l'Ecole normale supérieure*, 4e series, 1969, vol. 2, pp. 311-395. DOI: 10.24033/asens.1178

24. Besov O.V., Il'in V.P., Nikol'skii S.M. *Integral Representations of Functions and Imbedding Theorems*, V.H. Winston, 1978.

ОБРАТНАЯ ЗАДАЧА ВОССТАНОВЛЕНИЯ ПОТОКОВ ИЗ ИНТЕГРАЛЬНЫХ ДАННЫХ

С.Г. Пятков[1], *Е.И. Сафонов*[1], *А.А. Петров*[1]
[1]Югорский государственный университет, г. Ханты-Мансийск, Российская Федерация

В статье рассматриваются параболическое уравнение второго порядка и вопросы корректности обратных задач восстановления теплового потока на границе с использованием набора интегралов от решения с весами по области. Поток представим в виде конечного отрезка ряда с неизвестными коэффициентами, зависящими от времени. При определенных условиях на данные показано, что существует единственное решение задачи, которое непрерывно зависит от данных. Решение имеет все обобщенные производные, входящие в уравнение, суммируемые с некоторой степенью. Доказательство опирается на априорные оценки и принцип сжимающих отображений. Метод конструктивен и позволяет строить численные методы решения задачи. Численный алгоритм основан на методах конечных элементов и конечных разностей. Результаты численных экспериментов вполне удовлетворительны, а процедура построения решения устойчива при малых возмущениях.

Ключевые слова: обратная задача; граничная задача; параболическое уравнение; тепломассоперенос.

Литература

1. Алифанов, О.М. Обратные задачи исследования сложной теплопередачи / О.М. Алифанов, Е.А. Артюхин, А.В. Ненарокомов. – М.: Янус-К, 2009.

2. Пермяков, П.П. Восстановление граничных условий для моделирования для моделирования теплообмена на поверхности грунта / П.П. Пермяков, Т.А. Афанасьева, С.П. Варламов, П.Н. Скрябин // Арктика XXI век. Гуманитарные науки. – 2019. – Т. 17, № 1. – С. 27–35.

3. Пермяков, П.П. Идентификация параметров математической модели тепловлагопереноса в мерзлых грунтах / П.П. Пермяков. – Новосибирск: Наука, 1989.

4. Сабреков, А.Ф. Определение удельного потока метана из почвы с помощью обратного моделирования на основе сопряженных уравнений / А.Ф. Сабреков, М.В. Глаголев, И.Е. Терентьева // Доклады Международной конференции «Математическая биология и биоинформатика». – 2018. – Т. 7. – V. e94.

5. Белолипецкий, В.М. Оценка потока углерода между атмосферой и наземной экосистемой по измеренным на вышке вертикальным распределениям концентраций CO_2 / В.М. Белолипецкий, П.В. Белолипецкий // Вестник Новосибирского государственного университета. Серия: Информационные технологии. – 2011. – Т. 9, № 1. – С. 75–81.

6. Pyatkov, S.G. Determination of the Heat Transfer Coefficient in Mathematical Models of Heat and Mass Transfer / S.G. Pyatkov, V.A. Baranchuk // Mathematical Notes. – 2023. – V. 113, № 1. – P. 93–108.

7. Пятков, С.Г. Идентификация граничного условия в задачах тепломассопереноса / С.Г. Пятков, В.А. Баранчук // Челябинский физико-математический журнал. – 2022. – Т. 7, № 2. – С. 234–253.

8. Pyatkov, S.G. The Well-Posed Identification of the Interface Heat Transfer Coefficient Using an Inverse Heat Conduction Model / S.G. Pyatkov, A.A. Potapkov // Mathematics. – 2023. – V. 11, № 23. – P. 4739.

9. Kostin, A.B. On Some Problems of the Reconstruction of a Boundary Condition for a Parabolic Equation. II. / A.B. Kostin, A.I. Prilepko // Differential Equations. – 1996. – V. 32, № 11. – P. 1515–1525.

10. Kostin, A.B. On Some Problems of the Reconstruction of a Boundary Condition for a Parabolic Equation. I. / A.B. Kostin, A.I. Prilepko // Differential Equations. – 1996. – V. 32, № 1. – P. 113–122.

11. Pilant, M. An Iteration Method for the Determination of an Unknown Boundary Condition in a Parabolic Initial-Boundary Value Problem / M. Pilant, W. Rundell // Proceedings of the Edinburgh Mathematical Society. – 1989. – V. 32. – P. 59–71.

12. Hao Dinh Nho. Determination of the Heat Transfer Coefficients in Transient Heat Conduction / Nho Hao Dihn, Phan Xuan Thanh, D. Lesnik // Inverse Problems. – 2013. – V. 29, № 9. – P. 095020.

13. Hao Dinh Nho. Identification of Nonlinear Heat Transfer Laws from Boundary Observations / Dinh Nho Hao, Bui Veit Huong, Phan Xuan Thanh, D. Lesnik // Applicable Analysis. – 2014. – V. 94, № 9. – P. 1784–1799.

14. Kozhanov, A.I. Linear Inverse Problems for Some Classes of Nonlinear Nonstationary Equations / A.I. Kozhanov // Siberian Electronic Reports. – 2015. – V. 12. – P. 264–275.

15. Вержбицкий, М.А. О некоторых обратных задачах восстановления граничных режимов / М.А. Вержбицкий, С.Г. Пятков // Математические заметки СВФУ. – 2016. – Т. 23, № 2. – С. 3–18.

16. Triebel, H. Interpolation Theory. Function Spaces. Differential Operators / H. Triebel. – Berlin: Veb Deutscher Verlag des Wissenschaften, 1978.

17. Denk, R. Optimal L_p–L_q-Estimates for Parabolic Boundary Value Problems with Inhomogenous Data / R. Denk, M. Hieber, J. Pruess // Mathematische zeitschrift. – 2007. – V. 257, № 1. – P. 193–224.

18. Amann, H. Compact Embeddings of Vector-Valued Sobolev and Besov Spaces / H. Amann // Glasnik matematicki. – 2000. – V. 35, № 1. – P. 161–177.

19. Ladyzhenskaya, O.A. Linear and Quasi-Linear Equations of Parabolic Type / O.A. Ladyzhenskaya, V.A. Solonnikov, N.N. Ural'tseva. – Providence: AMS. – V. 23, 1968.

20. Behzadan A. Multiplication in Sobolev Spaces, Revisited. / A. Behzadan, M. Holst // Arkiv for Matematik. – 2021. – V. 59. – P. 275–306.

21. Meyries, M. Maximal Regularity with Temporal Weights for Parabolic Problems with Inhomogeneous Boundary Conditions / M. Meyries, R. Schnaubert // Mathematische nachrichten. – 2012. – V. 285, № 8–9. – P. 1032–1051.

22. Hummel, F. Elliptic and Parabolic Boundary Value Problems in Weighted Function Spaces / F. Hummel, N. Lindemulder // Potential Analysis. – 2022. – V. 57. – P. 601–669.

23. Grisvard, P. Equations Differentielles Abstraites / P. Grisvard // Annales scientifiques de l'Ecole normale supérieure, 4e series. – 1969. – V. 2. – P. 311–395.

24. Besov, O.V. Integral Representations of Functions and Imbedding Theorems / O.V. Besov, V.P. Il'in, S.M Nikol'skii. – Palm Beach: V.H. Winston, 1978.

Сергей Григорьевич Пятков, доктор физико-математических наук, профессор, Инженерная школа цифровых технологий, Югорский государственный университет (г. Ханты-Мансийск, Российская Федерация), s_pyatkov@ugrasu.ru.

Егор Иванович Сафонов, кандидат физико-математических наук, доцент, Инженерная школа цифровых технологий, Югорский государственный университет (г. Ханты-Мансийск, Российская Федерация), e_safonov@ugrasu.ru.

Алексей Аверьянович Петров, кандидат физико-математических наук, доцент, Инженерная школа цифровых технологий, Югорский государственный университет (г. Ханты-Мансийск, Российская Федерация), a_petrov@ugrasu.ru.