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HIERARCHICAL GAME UNDER UNCERTAINTY: A CASE OF DIFFERENT SOCIAL BEHAVIOR

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In the paper, a two-level hierarchical game with one upper-level player and N lower-level players is considered. The game models the heterogeneous social behavior of the bottom-level players. Unlike classical game-theoretic models, where all agents are strictly selfish, it is assumed that the bottom-level players are divided into two groups: committed altruists acting within the Berge equilibrium (the desire to maximize the payoff of others) and selfish agents adhering to the traditional Nash equilibrium and maximizing their own payoff. Furthermore, the payoff functions of each player are affected by interval uncertainty. The concept of a Pareto-guaranteed hybrid equilibrium is formalized for the proposed game. A linear-quadratic hierarchical game under uncertainty with one top-level player and three bottom-level players, two of whom are altruists and one of whom adheres to a selfish behavior model, is given as a model example. A Pareto-guaranteed hybrid equilibrium is constructed analytically for this game. The approach developed in this article allows for more accurate modeling of real-world socioeconomic processes characterized by information asymmetry and the presence of different behavioral incentives among individuals, for example, in macroeconomic governance systems or in models of state-society interactions.

Keywords: Berge equilibrium; Nash equilibrium; hybrid equilibrium; uncertainty; hierarchical game.

Introduction

The foundations of game theory as a mathematical science were laid back in the 1940s. From the very beginning, a significant difference between mathematical game theory and other mathematical sciences became apparent. In other branches of mathematics, the concept of a solution is clearly defined and uncontroversial. In mathematical game theory, there are numerous solution concepts, and researchers have yet to determine what constitutes a game solution. For example, for the non-cooperative game considered in this article, several solution concepts have been proposed.

The fact is that all proposed solution concepts simultaneously have both advantages and disadvantages. This makes it impossible to select one concept as definitively superior. The most successful solution concept in non-cooperative games is the Nash equilibrium. It was proposed by John Nash [1] in 1949. For this concept, Nash was awarded the Nobel Prize in Economics in 1994. Nash equilibrium is widely used in various fields of study, particularly in economics, sociology, political science, and military science. The reason for its popularity is its robustness. This is because the profile of Nash equilibrium strategies is robust to individual player deviations. That is, if any player attempts to deviate from the Nash equilibrium strategy, they cannot increase their payoff, but their payoff may decrease. Hundreds of articles devoted to Nash equilibrium appear annually in leading journals in operations research, game theory, and systems analysis.

However, "even the sun has spots". One of the shortcomings of Nash equilibrium is its "strongly selfish nature", corresponding to the desire of each participant in a conflict to maximize only their own benefit. Moreover, players completely disregard the interests of other players. The direct opposite of Nash equilibrium is the altruistic concept of Berge equilibrium. By applying this concept, each player makes every effort to increase the winnings of all other players, while forgetting about his own interests. The Berge equilibrium was formalised as a possible solution to the non-cooperative game by V.I. Zhukovsky in 1985 [2] during a critical review of a book by the French mathematician Claude Berge "Théorie générale des jeux a n personnes" [3]. In the 1980s V.I. Zhukovskiy started to investigate one of the conepts of equilibrium in normal form games introduced by Berge in [3]. In fact, Zhukovsky initially introduced this equilibrium as an alternative to the Nash equilibrium when a Nash equilibrium is absent from the game. However, subsequent research has shown that the Berge equilibrium is a rich concept, as it has multiple interpretations and reflects a variety of socioeconomic models of behavior in human interaction. Indeed, it can express the moral Golden Rule [4] "Behave to others as you would like them to behave to you", the principle of mutual support [5] or "positive" reciprocation, coordination, cooperation in non-cooperative settings and altruism.

In [6,7], Abalo and Kostreva give a more general definition of Berge equilibrium. Furthermore, in [6] a theorem of existence of Berge equilibrium that is also Nash equilibrium or Berge–Nash equilibrium is established for Berge equilibrium in sense Abalo and Kostreva. The existence of Berge–Nash equilibria was also studied in [8]. Note that in contrast to [6], where the Berge equilibrium is simultaneously a Nash equilibrium, this paper assumes a different approach. At the same time, the hybrid equilibrium considered in the paper (hereafter called Berge–Nash hybrid equilibrium) is not a Berge–Nash equilibrium in the sense of [6]. Nowadays, a large number of papers are devoted to the Berge equilibrium [4, 5, 9–12]. Note that there are papers that model the exact opposite behaviour [13,14]. But, the only explanation for this behaviour, in our opinion, is hatred of others (Anti-Berge) or self-hatred and masochism (Anti-Nash).

Real socioeconomic processes are highly complex and heterogeneous. Agents (players) operating in these systems may have completely different behavioral incentives. Therefore, applying either the Nash or Berge equilibrium concepts may not accurately describe the process model. In fact, some agents (possibly a small number in real systems) may exhibit altruistic behavior and adhere to the Berge equilibrium concept. Meanwhile, the remaining agents may adhere to the Nash equilibrium, seeking to maximize their own payoffs. In such a situation, a solution can be found in the concept of a hybrid equilibrium, which simultaneously takes into account the behavior of both groups of players. Such a solution will be called BN-hybrid equilibrium ($BNHE$).

Another problem arises in modeling and controlling complex socio-economic systems. This is the presence of a decision-making hierarchy. Such a hierarchical system can appear as a consequence of information "inequality". For example, in the model "tax service – people". The tax service, determining the amount of taxes, has information about how people will act at different levels of taxation. People, in forming their strategy, have no information about the specific decision of the tax service. In this case, the goal of the tax service is to choose the level of taxation that maximizes total tax collection. The construction of the hierarchical games theory was started in the works of Yu.B. Hermeyer [15]. The paper considers the case of a two-level hierarchy with one upper-level player and several players of the lower level of the hierarchy [16]. Players at the lower level of the hierarchy have different social preferences. Therefore, some of them have a selfish behavioural pattern, trying to maximize their own income by their choices. And other players are altruistic, they care about the good of other people. It is worth noting that there is always, perhaps, a small proportion of altruists in society.

Furthermore, it is assumed that the players' payoffs depend not only on their actions but also on uncertainty. Uncertainty has a non-stochastic character. Players know about it only the set of its possible values. Such problems were studied in [17,18]. Such uncertainty can be caused by external influences on the economic system, such as unpredictable exchange rate jumps, etc. Since players lack information regarding the probabilistic characteristics of these factors, it is assumed that they can only estimate the boundaries of their change.

The paper is structured as follows. Section 1 recalls the basic concepts of game theory. In Section 2, for a non-cooperative N-person game, the concept of $BNHE$ is formalized. Sufficient conditions for the existence of $BNHE$, reduced to the existence of a saddle point of a specially constructed Hermeyer convolution, are given. In Section 3, the concept of a Pareto-guaranteed hybrid equilibrium is formalised for a hierarchical game under uncertainty. Here it is assumed that the upper-level player seeks to maximize his payoff. Some of the lower level players adhere to the altruistic behavior model, and the rest adhere to the egoistic behavior model. In this case, all players have to take into account the impact of uncertainty, the realization of which they cannot influence. But uncertainty affects their payoff functions. The linear-quadratic version of the game is considered as an example.

1. Preliminaries

In this section, some concepts of game theory are recalled.

1.1. Non-Cooperative Game

Consider a non-cooperative N players game. Such a game is given by the tuple

$$\Gamma = \langle \mathbb{N}, \{X_i\}_{i\in\mathbb{N}}, \{f_i(x)\}_{i\in\mathbb{N}}\rangle, \tag{1}$$

where $\mathbb{N} = \{1, ..., N\}$ is the set of players' serial numbers; each player i chooses and applies his own pure strategy $x_i \in X_i \subset \mathbb{R}^{n_i}$, forming no coalition with the others, which induces a strategy profile $x = (x_1, ..., x_N) \in X = \prod\limits_{i\in\mathbb{N}} X_i \subset \mathbb{R}^n$ $(n = \sum\limits_{i\in\mathbb{N}} n_i)$; for each $i \in \mathbb{N}$ a payoff function $f_i(x)$ is defined on the strategy profile set X, which gives the payoff of player i. In addition, denote $(x\|z_i) = (x_1, ..., x_{i-1}, z_i, x_{i+1}, ...x_N)$ and $f = (f_1, ..., f_N)$.

The most popular concept of a solution for a non-cooperative game was proposed by Nash [1].

Definition 1. *The strategy profile $x^e = (x_1^e, \ldots, x_i^e, \ldots, x_N^e) \in X$ is called a Nash equilibrium in the game Γ if*

$$\max_{x_i\in X_i} f_i(x^e\|x_i) = f_i(x^e) \ \ (i \in \mathbb{N}).$$

In a Nash equilibrium, each player tends to maximise his own payoff. At the same time, acting selfishly, he does not think about the payoffs of the other players. Completely opposite are the motives of players using Berge equilibrium [2], where everyone directs all efforts to maximise all others' payoffs.

Definition 2. *The strategy profile $x^B = (x_1^B, \ldots, x_i^B, \ldots, x_N^B) \in X$ is called a Berge equilibrium in the game Γ if*

$$\max_{x\in X} f_i(x\|x_i^B) = f_i(x^B) \ \ (i \in \mathbb{N}).$$

1.2. Hierarchical Game

A two-level hierarchical game is considered in this paper. It is assumed that there is one upper-level player and several lower-level players. As usual [15, 16], player inequality is established by the right of first turn of the upper-level player. This is reflected in the information discrimination of lower-level players. Players of the lower-level are equal. Such a game can be associated with a tuple

$$\langle \mathbb{N}, U, \{X_i\}_{i\in\mathbb{N}}, g(u, x), \{f_i(u, x)\}_{i\in\mathbb{N}}\rangle, \tag{2}$$

where $\mathbb{N} = \{1, ..., N\}$ is the set of serial numbers of the lower-level players; U is the set of pure strategies of the upper-level player. His strategies are $u \in U \subset \mathbb{R}^{n_0}$. The set of strategies x_i of each i-th player of the lower-level is $X_i \subset \mathbb{R}^{n_i}$. $x = (x_1, ..., x_N) \in X = \prod\limits_{i\in\mathbb{N}} X_i \subset \mathbb{R}^n$ $(n = \sum\limits_{i\in\mathbb{N}} n_i)$. The scalar payoff functions of all players are defined on the Cartesian product $U \times X$. For the upper-level player this is $g(u, x)$, and for each i-th player in the lower-level, the payoff function is $f_i(u, x)$.

A game party is played as follows. In the first step, the upper-level player passes to the other players information about his set of strategies U. Based on this knowledge, lower-level players form their responses as functions of the form $x_i(u) : U \to X_i$ $(i \in \mathbb{N})$. Such functions are called counter-strategies in [19]. In forming $x_i(u)$, for every fixed pure strategy $u \in U$, the lower-level players use some optimality criterion, for example, a Nash equilibrium.

In the second step, lower-level players pass information about their chosen counter-strategies $x_i(u)$ to the upper-level player. With this information, the upper-level player chooses his strategy $u^* \in U$ to maximise his payoff $g(u, x(u)$. Namely, as a solution to the problem

$$g(u^*, x(u^*)) = \max_{u\in U} g(u, x(u)).$$

A solution so constructed is called a Stackelberg–Nash equilibrium [20].

1.3. Multi-Criteria Decision-Making Problem

Consider a multi-criteria decision-making problem (MCDM)

$$\Gamma_y = \langle Y, \{f_i(y)\}_{i=1,\ldots,N} \rangle.$$

In the Γ_y problem, the set of alternatives is $Y \subset \mathbb{R}^m$. The decision maker (DM) chooses an alternative $y \in Y$ such that the values of all N criteria $f_i(y)$ $(i = 1, \ldots, N)$ are reduced as much as possible. A Pareto-minimal solution to the problem Γ_y is [21] the alternative $y^P \in Y$ for which the system of N inequalities is incompatible

$$f_i(y^P) \geqslant f_i(y) \quad (i = 1, \ldots, N) \quad \forall y \in Y,$$

in which at least one inequality is strict. Pareto minimality means that the DM, deviating from y^P cannot decrease the value of any of the criteria without increasing the value of the other criterion.

2. BN-Hybrid Equilibrium

Consider a N-person noncooperative game (1), namely

$$\Gamma = \langle \mathbb{N}, \{X_i\}_{i \in \mathbb{N}}, \{f_i(x)\}_{i \in \mathbb{N}} \rangle.$$

Below we assume that all players in the game Γ are divided into two parts. Within these parts, players adhere to identical behavior patterns. These parts are not coalitions, as cooperation and interaction within each part are not implied.

So, consider that the set of players $\mathbb{N}$ in Γ is divided into two non-empty subsets. The first subset, denoted as $\mathbb{B}$, includes players who follow the Golden Rule of Morality ("Treat others as you would like others to treat you") in their decisions. Players of this set, choosing their strategy, adhere to the Berge equilibrium concept. Players who are not in the set $\mathbb{B}$ form the set $\mathbb{N} \backslash \mathbb{B}$. They hold the concept of Nash equilibrium, reflecting their desire to maximize their own payoff.

This situation is possible in a society where a part of its members are convinced altruists, they form the set $\mathbb{B}$, and the rest of the members of the society, who prioritize only themselves, form the set $\mathbb{N} \backslash \mathbb{B}$.

Below by $x_{\mathbb{B}}$ we denote the set of strategies x_i of all players i belonging to the set $\mathbb{B}$, namely

$$x_{\mathbb{B}} \in X_{\mathbb{B}} = \prod_{k \in \mathbb{B}} X_k.$$

For $j \in \mathbb{N} \backslash \mathbb{B}$ the set composed of strategies x_i of all players $i \in \mathbb{B}$ and strategies x_j of player j, denote by $x_{\mathbb{B}+j}$, and the whole set of such sets will be denoted as

$$X_{\mathbb{B}+j} = \prod_{k \in \mathbb{B}} X_k \times X_j.$$

Finally, for each player $i \in \mathbb{B}$, $x_{\mathbb{B}-i}$ will denote the set of strategies $x_{\mathbb{B}} \in X_{\mathbb{B}}$ with the strategy x_i of player i excluded from it, i.e.

$$x_{\mathbb{B}-i} \in X_{\mathbb{B}-i} = \prod_{k \in \mathbb{B} \backslash \{i\}} X_k.$$

Next, when $\mathbb{P} = \mathbb{B}$, $\mathbb{B} - i$ or $\mathbb{B} + j$, $(x \| z_{\mathbb{P}})$ will denote the strategy profile $x \in X$, in which the strategies of all players, in the subset $\mathbb{P}$ are replaced by their strategies taken from the set $z_{\mathbb{P}}$ generated from the strategy profile $z \in X$. We will also use the notation $N_{\mathbb{B}}$ for the number of players in the set $\mathbb{B}$.

In the following definition, the solution to the Γ game is not an equilibrium strategy profile, but a pair – the strategy profile x^* and the payoffs $f(x^*)$ delivered to it.

Definition 3. *The pair $(x^*, f(x^*))$ is called a BN-hybrid equilibrium (BNHE) of the game Γ if for all $x \in X$ the following conditions are satisfied*

$$\begin{aligned} \max_{x\in X} f_i(x^*\|x_{\mathbb{B}-i}) &= f_i(x^*) \quad (i \in \mathbb{B}), \\ \max_{x\in X} f_j(x^*\|x_{\mathbb{B}+j}) &= f_j(x^*) \quad (j \in \mathbb{N}\backslash\mathbb{B}). \end{aligned} \tag{3}$$

Note that the first condition in (3) means that in $BNHE$ the win function of each player i in the set $\mathbb{B}$ is maximized by all players in this set except for player i himself. The meaning of the second condition in (3) is that for each player who is not in the set $\mathbb{B}$, his payoff function is maximized both by himself and by all players in the set $\mathbb{B}$.

Remark 1. Equalities (3) mean that $BNHE$ is stable both with respect to the deviation from it of an individual player not in the altruistic subset $\mathbb{B}$, and with respect to the deviation from it of all players in the subset $\mathbb{B}$.

Remark 2. As follows from Definition 3, when the subset $\mathbb{B}$ consists of only one player i, the first condition in (3) is absent. In this case, the value of the function $f_i(x)$ is not considered in the construction of $BNHE$.

Remark 3. If exactly one player j from the set $\mathbb{N}$ of all players of the game Γ does not belong to the subset of altruistic players $\mathbb{B}$, then to construct $BNHE$ it is enough:

First, to find the strategy profile $x^* \in X$ from the condition

$$\max_{x\in X} f_j(x) = f_j(x^*). \tag{4}$$

Second, check for each $i \in \mathbb{B}$ and the strategy profile x^* the validity of the equality

$$\max_{x\in X} f_i(x^*\|x_{\mathbb{B}-i}) = f_i(x^*). \tag{5}$$

The strategy profile x^* obtained from (4) and satisfying (5) for all $i \in \mathbb{B}$, exactly forms $BNHE$ $(x^*, f(x^*))$. The above approach can be demonstrated by the following example.

Example 1. Consider Bryant's game [22] with three players, where players 1 and 2 form the set $\mathbb{B}$ of altruistic players. The strategy of each player i is to choose a real number between 0 and 5, that is, $x_i \in [0, 5]$ $(i = 1, 2, 3)$. On the set $X = [0, 5] \times [0, 5] \times [0, 5]$ the players' payoff functions

$$f_i(x) = 2\min\{x_1, x_2, x_3\} - x_i \quad (i = 1, 2, 3)$$

are defined.

Note that Nash equilibria in this game has the form [22]

$$X^e = \{x^e = (\lambda, \lambda, \lambda) \mid \forall\lambda \in [0, 5]\}$$

the Berge equilibria takes exactly the same form. Following the method proposed in Remark 3, first, the strategy profile x^* is found from the condition

$$\max_{x\in X} f_3(x) = f_3(x^*).$$

It is easy to see that the maximum value of the function $f_3(x) = 2\min\{x_1, x_2, x_3\} - x_3$ on X is reached only when $x^* = (5, 5, 5)$. Second, let us verify that the conditions (5) for $i \in \{1, 2\}$ and $x^* = (5, 5, 5)$ are true, namely, that the maximum value of the function at $x_1, x_2 \in [0, 5]$

$$f_1(5, x_2, 5) = 2\min\{5, x_2\} - 5 = 2x_2 - 5$$

and

$$f_2(x_1, 5, 5, 5) = 2\min\{x_1, 5\} - 5 = 2x_1 - 5$$

are reached at $x_2^* = 5$ and $x_1^* = 5$, respectively. Hence the strategy profile $x^* = (5, 5, 5)$ and the payoffs $f(x^*) = (5, 5, 5)$ form $BNHE$.

Sufficient conditions for the existence of $BNHE$ are constructed using the approach proposed in [23] and applied in [4, 17]. For this purpose, we introduce the N-vector $z = (z_1, \ldots, z_N) \in X$ and the Hermeyer convolution [15, 24], where $i = 1, \ldots, N$,

$$\begin{gathered} \varphi_i(x,z) = f_i(z\|x_{\mathbb{B}-i}) - f_i(z) \ \ (i \in \mathbb{B}), \\ \varphi_j(x,z) = f_j(z\|x_{\mathbb{B}+j}) - f_j(z) \ \ (i \in \mathbb{N} \setminus \mathbb{B}), \end{gathered} \tag{6}$$

$$\psi(x,z) = \max_{r=1,\ldots,N} \varphi_r(x,z). \tag{7}$$

The saddle point $(x^0, z^*) \in X \times X$ of the scalar function $\psi(x,z)$ of (6),(7) is defined by a chain of inequalities

$$\psi(x,z^*) \leq \psi(x^0,z^*) \leq \psi(x^0,z) \ \ \forall x \in X, z \in X. \tag{8}$$

Theorem 1. *If in the zero-sum game*

$$\Gamma_a = \langle X, Z = X, \psi(x,z) \rangle$$

the saddle point (x^0, z^*) is found, then the maximin strategy $z^* \in X$ is the $BNHE$ of the game Γ.

*Proof.*If we put $z = x^0$ in (8), then from (6) and (7) it follows that $\psi(x^0, x^0) = 0$, hence (by transitivity) we obtain $\psi(x, z^*) \leq 0 \quad \forall x \in X$. Then in consequence of $\max\limits_{r=1,\ldots,N} \varphi_r(x,z^*) \leq 0$ $\forall x \in X$ and (6) N inequalities

$$\begin{gathered} f_i(z\|x_{\mathbb{B}-i}) \leqslant f_i(z) \quad (i \in \mathbb{B}) \quad \forall x_{\mathbb{B}-i} \in X_{\mathbb{B}-i}, \\ f_j(z\|x_{\mathbb{B}+j}) \leqslant f_j(z) \quad (i \in \mathbb{N} \setminus \mathbb{B}) \quad \forall x_{\mathbb{B}+j} \in X_{\mathbb{B}+j} \end{gathered}$$

are satisfied.

The first $N_{\mathbb{B}}$ of these inequalities, by virtue of (6), means that the first condition in (3) is true for the strategy profile $z^* \in X$ in the Γ. The second group, consisting of $N - N_{\mathbb{B}}$ inequalities, realizes for the strategy profile z^* the second condition from (3).

□

Remark 4. According to Theorem 1, the construction of $BNHE$ x^* reduces to finding the saddle point (x^0, z^*) of the Hermeyer convolution $\psi(x,z)$ of (6), (7). Exactly, we obtain the following constructive way of constructing the $BNHE$ of the game Γ:
1) specify a non-empty subset $\mathbb{B}$ of players using altruistic behavior;
2) determine by formulas (6),(7) the scalar function $\psi(x,z)$;
3) find the saddle point (x^0, z^*) of the function $\psi(x,z)$ (satisfying the chain of inequalities (8));
4) construct the values of N scalar functions $f_i(z^*)$ ($i \in \mathbb{N}$).

Then the pair $(z^*, f(z^*) = (f_1(z^*), \ldots, f_N(z^*))$ exactly forms a BN-hybrid equilibrium in the Γ game: each player $i \in\in \mathbb{N}$ should use his strategy z_i^* from the strategy profile z^*, thus ensuring himself a payoff $f_i(z^*)$.

The existence of BN-hybrid equilibrium in mixed strategies was proved in [25].

3. Hierarchical Game under Uncertainty

In this section, a hierarchical two-level game is considered. This game is defined by the tuple

$$\langle \mathbb{N}, U, \{X_i\}_{i \in \mathbb{N}}, Y, g(u,x,y), \{f_i(u,x,y)\}_{i \in \mathbb{N}} \rangle. \tag{9}$$

In (9), just as in game (2), one upper-level player and N players of the lower-level. The set of ordinal numbers of the lower-level players $\mathbb{N}$, the set of strategies of the upper-level player $U \subset \mathbb{R}^{n_0}$ and the set of lower-level players strategies $X_i \subset \mathbb{R}^{n_i}$ ($i \in \mathbb{N}$) are the same as in game (2).

Unlike the game (2), the game (9) contains a set $Y \subset \mathbb{R}^m$ of uncertainties y. In the decision-making process, all players have to take into account not only the rationality of other players' choice of other players, but also that one of the values of uncertainty y is realised. What specific value of uncertainty is realised the players are not informed about. They also have no information about the probability distributions of uncertainty. Players know only the set of possible values of uncertainty Y, of which only one is realised in the game.

After all players choose their strategies and realise the uncertainty, a triple (u, x, y) is formed, where $x = (x_1, \ldots, x_N)$ is the strategy profile of the lower-level players. On the set of all such triples

$$U \times X \times Y = U \times X_1 \times \ldots \times X_N \times Y \subset \mathbb{R}^{n_0+n_1+\ldots+n_N+m}$$

the payoff functions of the players of the second level

$$f_i(u, x, y) : U \times X \times Y \to \mathbb{R} \quad (i = 1, \ldots, N),$$

and the payoff function of the upper-level player

$$g(u, x, y) : U \times X \times Y \to \mathbb{R}$$

are defined.

The value of the payoff function on a certain realised triple (u, x, y) is the payoff of the relevant player. Next, consider that all lower-level players (as in Section 2) are divided into two non-empty non-intersecting sets. The first set is the set of altruists. Each player of this set, choosing his strategy, thinks about the welfare of all others. At the same time, he does not care about his own payoff. Such players in their actions adhere to the concept of Berge equilibrium. Let us denote, as in Section 2, the set of these players by $\mathbb{B}$. The second set is formed by selfish players. They choose a strategy that maximises their own payoff and do not care about the other players. Such players, making their choice, adhere to the concept of Nash equilibrium. The set of such players is $\mathbb{N} \setminus \mathbb{B}$. A party of game (9) develops as follows.

In the first step, the upper-level player passes to the lower-level players information about the set U of all his strategies u. Then, based on the knowledge of U, the lower-level players, given their social position, form their responses in the counter-strategy class

$$x_i(u, y) \quad (i = 1, \ldots, N), \quad \forall u \in U, \ y \in Y. \tag{10}$$

This results in the vector function $x(u, y) = (x_1(u, y), \ldots, x_N(u, y))$.

In the second step of the game, players at the lower-level of the hierarchy pass information to the upper-level player about the counter-strategies they have formed (10). Substituting (10) into his payoff function $g(u, x, y)$, the upper-level player obtains $g(u, x(u, y), y)$, depending only on his strategy u and uncertainty y. After this, he chooses his strategy $u^* \in U$, seeking to maximise his payoff $g(u^*, x(u^*, y), y)$ if possible. In doing so, he is forced to focus on the possibility of realising any uncertainty $y \in Y$.

Further, at the moment of choosing u^*, some particular value of uncertainty $y^* \in Y$ is realised. After that, the upper-level player determines his payoff $g(u^*, x(u^*, y^*), y^*)$. And lower-level players, based on their knowledge of u^* and y^*, calculate their payoffs $f_i(u^*, x(u^*, y^*), y^*)$ $(i = 1, \ldots, N)$.

We believe that to account for uncertainty, players should follow Hermeyer's principle of greatest guaranteed outcome [15]. Namely, expect to realise the "baddest" uncertainty for them. That is, uncertainty that delivers some vector minimum, for example, a Pareto minimum [21].

Formalising the solution, we will follow the approach proposed in [18] and called "analogue of vector saddle point". Here the solution is constructed as a vector saddle point, in which the maximum is replaced by a Stackelberg–Nash equilibrium (taking into account the social position of the players) in the game obtained from (9) under fixed uncertainty y. And the minimum is replaced by a Pareto minimum in the multi-criteria problem obtained from (9) for a fixed profile of strategies (u, x).

Definition 4. *We will call the quadratic (u^*, x^*, g^*, f^*) a Pareto-guaranteed hybrid equilibrium (PGHE) in a two-level hierarchical game* (9), *with different social behaviour of the players at the lower-level of the hierarchy, if there exists an uncertainty $y^P \in Y$ such that*

1) *the strategy of the upper-level player u^* satisfies the condition*

$$g(u^*, x(u^*, y^P), y^P) = \max_{u \in U} g(u, x(u, y^P), y^P),$$

where

$$f_i(u,x(u,y^P),y^P) = \max_{x\in X} f_i(u,x(u,y^P)\|x_{\mathbb{B}-i},y^P) \quad \forall u\in U \quad (i\in\mathbb{B}),$$
$$f_j(u,x(u,y^P),y^P) = \max_{x\in X} f_j(u,x(u,y^P)\|x_{\mathbb{B}+j},y^P) \quad \forall u\in U; \quad (j\in\mathbb{N}\backslash\mathbb{B});$$

2) *a strategy profile of second-level players is*

$$x^* = (x_1^*, x_2^*, \ldots, x_N^*),$$

where

$$x_i^* = x(u^*, y^P) \quad (i=1,\ldots,N);$$

3) *for any uncertainty $y\in Y$ a system of $N+1$ inequalities*

$$\begin{array}{l} g(u^*,x^*,y^P) \;\geqslant\; g(u^*,x(u^*,y),y), \\ f_i(u^*,x*,y^P) \;\geqslant\; f_i(u^*,x(u^*,y),y) \quad (i=1,\ldots,N), \end{array} \tag{11}$$

where at least one of the inequalities is strict, is inconsistent.

In this case, the value $g^* = g(u^*,x^*,y^P)$ will be called the guarantee of the upper-level player, and the vector

$$f^* = (f_1^*, f_2^*, \ldots, f_N^*),$$

where

$$f_i^* = f_i(u^*,x^*,y^P) \quad (i=1,\ldots,N),$$

will be called the vector of guarantees of the lower-level players.

Remark 5. Let

$$f_{\sum}(u,x,y) \;=\; g(u,x,y) + \sum_{i=1}^{N}(u,x,y).$$

For any uncertainty $y^P\in Y$ found from the condition

$$f_{\sum}(u^*,x(u^*,y^P),y^P) \;=\; \min_{y\in Y}(u^*,x(u^*,y),y), \tag{12}$$

will be [21] an inconsistent system of inequalities (11), of which at least one inequality is strict.

Remark 6. The meaning of the guarantees g^* and $f^* = (f_1^*, f_2^*, \ldots, f_N^*)$ is that when players choose their strategy profile (u^*,x^*) and realising any uncertainty $y\in Y$, they cannot obtain payoffs that are smaller than the guarantees for all players at once.

Next, an example of constructing $PGHE$ in a linear-quadratic game is given.

Example 2.

Consider the game (9), where

$$g(u,x,y) = 2x_1u + 2x_2u + 2x_3u - 55u^2 + y^2 - 2x_2y + 14y, \tag{13}$$

$$\begin{aligned} f_1(u,x,y) = {} & -5x_1^2 - x_2^2 - x_3^2 + 4x_1x_2 + 2x_1x_3 - x_2x_3 + y^2 - \\ & -2x_1y + 2x_2y - 4x_3y + 2x_1u + 4x_2u + 6x_3u, \\ f_2(u,x,y) = {} & x_1^2 + x_2^2 - \tfrac{x_3^2}{2} + 2x_1x_2 + 4x_1x_3 - 2x_2x_3 + 2y^2 + \\ & +2x_1y + 2x_2y + 2x_3y + 2x_1u + 2x_2u + 4x_3u, \\ f_3(u,x,y) = {} & x_1^2 - \tfrac{x_2^2}{2} + x_3^2 + 2x_1x_2 - 2x_1x_3 - 2x_2x_3 + 3y^2 + \\ & +2x_1y - 4x_2y + 2x_3y + 4x_1u + 6x_2u - 2x_3u, \end{aligned}$$

and $u\in R$, $x=(x_1,x_2,x_3)\in R^3$, uncertainty $y\in R$.

In this game, three players are in the lower level of the hierarchy. The first player is selfish. Players 2 and 3 are altruists. Their strategies are x_1, x_2 and x_3, respectively.

According to Remark 3. First, the strategy profile $x(u, y) = (x_1(u, y), x_2(u, y), x_3(u, y))$ can be found from the condition

$$f_1(u, x(u, y), y) = \max_{x \in R^3} f_1(u, x, y) \quad \forall u \in R,\ \forall y \in R. \tag{14}$$

Equality (14) is satisfied if

$$\begin{aligned} \left.\frac{\partial f_1(u,x,y)}{\partial x_1}\right|_{x=x(u,y)} &= -10x_1 + 4x_2 + 2x_3 - 2y + 2u = 0, \\ \left.\frac{\partial f_1(u,x,y)}{\partial x_2}\right|_{x=x(u,y)} &= 4x_1 - 2x_2 - x_3 + 2y + 4u = 0, \\ \left.\frac{\partial f_1(u,x,y)}{\partial x_3}\right|_{x=x(u,y)} &= 2x_1 - x_2 - 2x_3 - 4y + 6u = 0, \end{aligned} \tag{15}$$

and

$$\frac{\partial^2 f_1(u, x, y)}{\partial x_i \partial x_j} < 0. \tag{16}$$

The system of linear equations (15) has a unique solution

$$x_1(u, y) = \frac{41u - 15y}{3}, \quad x_2(u, y) = \frac{32u + 14y}{3}, \quad x_3(u, y) = \frac{8u - 10y}{3}. \tag{17}$$

The Hessian matrix is

$$\frac{\partial^2 f_1(u, x, y)}{\partial x_i \partial x_j} = \begin{pmatrix} -10 & 4 & 2 \\ 4 & -2 & -1 \\ 2 & -1 & -2 \end{pmatrix}. \tag{18}$$

According Sylvester's criterion, the Hessian matrix (18) is negative-definite (i.e. (16) is true). Therefore, the lower-level players' strategies (17) satisfy condition (14).

From

$$\frac{\partial f_2(u, x, y)}{\partial x_3} = \frac{\partial f_1(u, x, y)}{\partial x_2} = 4x_1 - 2x_2 - x_3 + 2y + 4u,$$

and

$$\frac{\partial^2 f_2(u, x, y)}{\partial x_3^2} = -1 < 0$$

it follows that $x_3(u, y)$ found in (17) is a solution to problem

$$f_2(u, x_1, x_2, x_3(u, y), y) = \max_{x_3 \in R} f_2(u, x_1, x_2, x_3, y) \quad \forall u \in R,\ \forall y \in R.$$

Similarly, according to

$$\frac{\partial f_3(u, x, y)}{\partial x_2} = \frac{\partial f_1(u, x, y)}{\partial x_3} = 2x_1 - x_2 - 2x_3 - 4y + 6u,$$

and

$$\frac{\partial^2 f_3(u, x, y)}{\partial x_2^2} = -1 < 0,$$

$x_2(u, y)$ is a solution to problem

$$f_3(u, x_1, x_2(u, y), x_3, y) = \max_{x_2 \in R} f_2(u, x_1, x_2, x_3, y) \quad \forall u \in R,\ \forall y \in R.$$

Next, substituting (17) into (13), we get

$$g(u, x(u, y), y) = -u^2 - \frac{86}{3}uy - \frac{25}{3}y^2 + 14y.$$

The upper-level player maximizes his payoff function g(u,x(u,y),y) by choosing his strategy u^* such that

$$g(u^*, x(u^*, y^P), y^P) = \max_{u \in R} g(u, x(u, y^P), y^P).$$

From

$$\left.\frac{\partial g(u, x(u, y^P), y^P)}{\partial u}\right|_{u=u^*} = -2u^* - \tfrac{86}{3} y^P = 0, \qquad \left.\frac{\partial^2 g(u, x(u, y^P), y^P)}{\partial u^2}\right|_{u=u^*} = -2 < 0,$$

it follows

$$u^* = -\frac{43}{3} y^P. \tag{19}$$

Substituting (19) into (17), we get

$$\begin{gathered} x_1^* = x_1(u^*, y^P) = -\tfrac{1808 y^P}{9}, \quad x_2^* = x_2(u^*, y^P) = -\tfrac{1334 y^P}{9}, \\ x_3^* = x_3(u^*, y^P) = -\tfrac{374 y^P}{9}. \end{gathered} \tag{20}$$

At the same time, according to remark 5, uncertainty y^P is found from the equality (12), where

$$\begin{gathered} f_{\sum}(u, x, y) = (u, x, y) + f_1(u, x, y) + f_2(u, x, y) + f_3(u, x, y) = \\ = -3x_1^2 - \tfrac{x_2^2}{2} - \tfrac{x_3^2}{2} + 8x_1 x_2 + 4x_1 x_3 - 5x_2 x_3 + \\ +7y^2 + 2x_1 y + 14y + 10x_1 u + 14x_2 u + 10x_3 u - 55u. \end{gathered}$$

Hence,

$$f_{\sum}(u^*, x^*, y) = -3(x_1^*)^2 - \frac{(x_2^*)^2}{2} - \frac{(x_3^*)^2}{2} + 8x_1^* x_2^* + 4x_1^* x_3^* - 5x_2^* x_3^* +$$

$$+7y^2 + 2x_1^* y + 14y + 10x_1^* u^* + 14x_2^* u^* + 10x_3^* u^* - 55u^*.$$

From

$$\left.\frac{\partial f_{\sum}(u^*, x^*, y)}{\partial y}\right|_{y=y^P} = 14y^P + 14 = 0$$

and

$$\left.\frac{\partial^2 f_{\sum}(u^*, x^*, y)}{\partial y^2}\right|_{y=y^P} = 14 > 0$$

implies that $\max_y f_{\sum}(u^*, x^*, y)$ is achieved at $y^P = -1$.

Substituting y^P into (27) and (28), we obtain

$$u^* = \frac{43}{3}, \quad x_1^* = \frac{1808}{9}, \quad x_2^* = \frac{1334}{9}, \quad x_3^* = \frac{374}{9}.$$

To complete the construction of $PGHE$, let's calculate the guarantees

$$\begin{gathered} g^* = g(u^*, x^*, y^P) = \frac{1648}{9}, \qquad f_1^* = f_1(u^*, x^*, y^P) = -\frac{6296435}{81}, \\ f_2^* = f_1(u^*, x^*, y^P) = \frac{12449656}{81}, \quad f_3^* = f_1(u^*, x^*, y^P) = \frac{6870433}{81}. \end{gathered}$$

So, $PGHE$ has the form (u^*, x^*, g^*, f^*), where $u^* = \dfrac{43}{3}$, $x = \left(\dfrac{1808}{9}, \dfrac{1334}{9}, \dfrac{374}{9}\right)$, $g^* = \dfrac{1648}{9}$, $f^* = (f_1^*, f_2^*, f_3^*) = \left(-\dfrac{6296435}{81}, \dfrac{12449656}{81}, \dfrac{6870433}{81}\right)$.

Discussion

The behaviour of agents in complex socio-economic systems does not always correspond to one model of behaviour. Often in one system there are agents with different social behaviour. Thus, the paper attempts to describe a system in which there are both selfish and altruistic agents. In addition, the presence of hierarchy in the system and the presence of uncertain factors were taken into account.

The authors see the development of the work, firstly, in the combination of the considered models of behaviour with other models. Thus, some agents may adhere to concept of Kantian equilibrium [26], or to concept of weak Berge equilibrium [27].

Second, when formalising a Pareto-guaranteed hybrid equilibrium, it was assumed that players construct a solution by counting on the emergence of a "worst-for-all at once" (Pareto minimum) uncertainty. However, another approach is possible. Each player counts on the worst uncertainty for him personally. And based on this assumption, a decision is already made. The guarantees obtained in this way could be called strong guarantees, and the equilibrium could be called a strongly guaranteed equilibrium. Strong guarantees will be the lowest, and since each agent counts on his "bad" uncertainty, none of the players will receive a payoff less than his guarantee.

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ИЕРАРХИЧЕСКАЯ ИГРА ПРИ НЕОПРЕДЕЛЕННОСТИ: СЛУЧАЙ РАЗЛИЧНОГО СОЦИАЛЬНОГО ПОВЕДЕНИЯ

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В статье рассматривается двухуровневая иерархическая игра с одним игроком на верхнем уровне иерархии и N игроками на нижнем уровне. В игре моделируется гетерогенное социальное поведения игроков нижнего уровня. В отличие от классических теоретико-игровых моделей, где все агенты строго эгоистичны, предполагается, что игроки нижнего уровня разделены на две группы: убежденных альтруистов, действующих в рамках концепции равновесия по Бержа (стремление максимизировать выигрыш окружающих), и эгоистичных агентов, придерживающихся традиционного равновесия по Нэшу и максимизарующих собственный выигрыш. Кроме того, на функции выигрыша каждого из игроков влияет интервальная неопределенность. Для предложенной игры формализовано понятие гарантированного по Парето гибридного равновесия. В качестве модельного примера приведена линейно-квадратичная иерархическая игра при неопределенности с одним игроком верхнего уровня, и тремя игроками нижнего, два из которых являются альтруистами, а один придерживается эгоистической модели поведения. Для данной игры в аналитическом виде построено гарантированное по Парето гибридное равновесие. Разработанный в статье подход позволяет более точно моделировать реальные социально-экономические процессы, характеризующиеся информационной асимметрией и наличием разных поведенческих стимулов у людей, например, в макроэкономических системах управления или в моделях взаимодействия государства и общества.

Ключевые слова: равновесие по Бержу; равновесие по Нэшу; гибридное равновесие; неопределенность; иерархическая игра.

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