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COMBINED RUNGE–KUTTA AND PIECEWISE CONSTANT ARGUMENT METHODS FOR NONLINEAR SYSTEMS OF DIFFERENTIAL EQUATIONS

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The applications of the piecewise constant argument method combined with the method are developed for solving first-order nonlinear systems of ordinary differential equations. Using the coefficients of a seventh-order method, a differential equation with piecewise constant arguments is constructed, corresponding to the considered initial value problem and depending on a positive integer N. It is proved that this equation has a unique piecewise-smooth solution, which serves as an approximate solution to the initial value problem for large N. The efficiency of the proposed method is demonstrated through several numerical examples, showing higher accuracy and faster convergence compared to the higher-order Haar wavelet collocation method and other methods reported in the literature.

Keywords: linear ordinary differential equation; piecewise constant argument; approximate solution; method; absolute error.

Introduction

Differential equations form the basis of many physical systems, prominently appearing in fields such as engineering, population dynamics, applied mathematics, physics, economics, and astronomy [1]. Zill [2] illustrates their application in modeling phenomena like bacterial growth, radioactive decay, fossil aging, Newton's law of cooling, and salt solution mixing.

Systems of ODEs are similarly vital in various scientific and engineering contexts, such as tank mixing problems, electrical networks, and mass-spring systems [3]. Numerous numerical techniques have been developed to solve these systems, including the differential transform approximation [4] and Haar wavelets collocation methods [5]. Aziz et al. [6] introduced a new numerical technique using Haar wavelets for nonlinear integral-type equations.

Alternative numerical methods like the finite element approach have been widely applied to nonlinear equations [7]. Both mesh-based and meshless methods have advanced the solving of linear and nonlinear ODEs, PDEs, and fractional differential equations [8, 9].

Nonnegativity of solutions and global unique existence of two systems of nonlinear first-order ODEs modeling Ca^{2+} is studied [10]. A finite-time stability for hybrid dynamical systems with deviating argument is explored using a hybrid control scheme in [11]. The reliability of the Non-Fixed Step-Size Algorithm offering efficient integration over varying intervals for stiff differential systems is confirmed through validation of its properties [12]. Finding exact periodic solutions of first-order nonhomogeneous differential equations with piecewise constant arguments, providing conditions and explicit solutions are given in [13].

In a recently published paper [5], a new Haar wavelet collocation method was developed for solving linear and nonlinear systems of ordinary differential equations. The stability and convergence of the technique were studied in detail. An approximate solution for nonlinear systems of ordinary differential equations was also presented in [14] using a variable step-size formulation of a Simpson-type second-derivative block method.

We consider two system of equations

$$\begin{cases} y_1' = f_1(t, y_1, y_2), \\ y_2' = f_2(t, y_1, y_2), \end{cases} \qquad t \in [a, b] \tag{1}$$

with initial conditions

$$y_1(a) = y_{10}, \quad y_2(a) = y_{20},$$

where $f_i, \ i = 1, 2$ are real valued differentiable functions are continuous functions on $[a, b] \times \mathbb{R}$, and y_{10} and y_{20} are real constants.

We introduced a system of ordinary differential equations with piecewise constant arguments corresponding to the considered initial value problems, which depend on a positive integer number N. It is proved that this system of equations has a unique piecewise-smooth solution, which will be an approximate solution for the considered initial value problem for large N. Numerical results are given on examples showing the efficiency and high accuracy of the suggested method. The efficiency of the methods is illustrated with certain numerical examples, for which is more accurate with faster convergence.

1. Differential Equations with Piecewise Constant Arguments

Let $(y_1(t), y_2(t))$ be solution of the initial value problem (1) and $|y_i(t)| \le d$, $i = 1, 2$. Further, we suppose that the functions $f_i(t, y_1, y_2)$ has the compact support $D = [a, b] \times [-2d, 2d]^2$, i.e. we assume

$$f_i(t, y_1, y_2) = 0 \quad \text{for} \quad (t, y_1, y_2) \notin D, \quad i = 1, 2. \tag{2}$$

We introduce a system of differential equations with piecewise constant arguments corresponding to the nonlinear differential equation (1). Let n be a positive integer, and define $t_k = kh+a$, where $h = \frac{b-a}{n}$ and $k = 0, 1, 2, \dots, n$. For $t \in [t_k, t_{k+1})$ with $k = 0, 1, 2, \dots, n$, consider the following differential equation with piecewise constant arguments.

This study focuses on optimizing the coefficients of a seventh-order method comprising thirteen stages for solving initial value problems. Given the structure of the method, which involves thirteen stages, a corresponding set of thirteen free parameters is employed during the training process [15].

The numerical approximation to the solution of (1) using the classical seventh-order method is given by:

$$\tfrac{y_{ik+1} - y_{ik}}{h} = K_i(t_k), \quad y_i(t_0) = y_{i0}, \quad t \in [t_k, t_{k+1}), \quad k = 0, 1, 2, \dots, n-1. \tag{3}$$

where the increment function $K_i(t_k)$ is computed as a weighted sum of the intermediate stage derivatives [16]:

$$K_1(t_k) = y_{1k} + \tfrac{31}{720}k_1 + \tfrac{16}{75}k_6 + \tfrac{16807}{79200}k_7 + \tfrac{16807}{79200}k_8 + \tfrac{243}{1760}k_9 + \tfrac{243}{1760}k_{12} + \tfrac{31}{720}k_{13},$$

$$K_2(t_k) = y_{2k} + \tfrac{31}{720}l_1 + \tfrac{16}{75}l_6 + \tfrac{16807}{79200}l_7 + \tfrac{16807}{79200}l_8 + \tfrac{243}{1760}l_9 + \tfrac{243}{1760}l_{12} + \tfrac{31}{720}l_{13}.$$

For any given positive integer n, we define an auxiliary differential equation with a piecewise constant argument that corresponds to the nonlinear differential equation in (1) as follows:

$$y_i'(t) = F_i(t), \quad y_i(t_0) = y_{i0}, \quad i = 1, 2, \quad t \in (a, b), \tag{4}$$

where

$$F_i(t) = K_i(t_k) \quad \text{for} \quad t \in [t_k, t_{k+1}), \quad k = 0, 1, 2, \dots, n-1,$$

the values y_k are defined by

$$y_{k+1} = y\big(t_{k+1}\big) = \lim_{t \to t_{k+1}-0} \int_{t_k}^{t} F_i(s)ds = K_i(t_k)\frac{b-a}{n}.$$

Due to the construction of $K_i(t_k)$, it satisfies the following asymptotic consistency condition:

$$K_i(t_k) - f_i(t_k, y_{1k}, y_{2k}) \to 0 \quad \text{as} \quad n \to \infty. \tag{5}$$

A solution of (4) is defined as follows [13].

Definition 1. *A vector function $(y_1(t), y_2(t))$ is called a solution of the initial value problem (4) if the following conditions are satisfied:*

(i) $y_1(t)$, $y_2(t)$ are continuous on the interval $[a, b]$;

(ii) $y_1'(t)$, $y_2'(t)$ *exist and continuous on* $[a,b]$ *with possible exception at points* $t = t_{k+1}$, *where* $k = 0, 1, \ldots, n-1$, *at which the one-sided derivatives exist;*

(iii) *the function* $y_1(t)$, $y_2(t)$ *satisfies the differential equation in (4) for all* $t \in (a,b)$, *possible excluding the points* $t = t_{k+1}$, *where* $k = 0, 1, \ldots, n-1$.

Theorem 1. *For any positive integer number n, the initial value problem (4) has a unique solution* $(y_1(t), y_2(t)) := (y_{1n}(t), y_{2n}(t))$, *as defined in (8).*

We observe that, under the assumption stated in (2), the functions $f_i(t, y_1, y_2)$, $i = 1, 2$ are bounded. Consequently, the sequence of functions $y_{1n}(t)$ and $y_{2n}(t)$, defined in (8), are uniformly bounded; that is, there exists a positive constant J such that

$$|y_{in}(t)| \leq J, \quad \text{for} \quad \text{all} \quad t \in [a,b], \quad n \quad \text{and} \quad i = 1, 2.$$

The following theorem claims that the vector (y_{1n}, y_{2n}) can be approximated solution for the initial value problem (1) for large n.

Theorem 2. *For any given* $\varepsilon > 0$, *there exists a positive integer* $n_0 = n(\varepsilon)$ *such that for all integers* $n > n_0$, *the following inequality*

$$\sup_{t \in [a,b]} |y_{in}'(t) - K_i(t)| < \varepsilon, \quad i = 1, 2$$

holds, where (y_{1n}, y_{2n}) *denotes the solution of the initial value problem (4).*

2. Proof of the Main Results

Proof of the Theorem 1. Let n be a positive integer number. For $t \in [t_0, t_1)$ the equation (4) is

$$y_i'(t) = K_i(t_0).$$

Integrating these equations, we have

$$y_i(t) = K_i(t_0)(t - t_0) + y_i(t_0), \qquad i = 1, 2, \qquad t \in [t_0, t_1). \tag{6}$$

Since $y_i(t), i = 1, 2$ is continuous $[t_0, t_1]$, there exist the following limits

$$y_i(t_1) = \lim_{t \to t_1 - 0} y_i(t).$$

Let $y = (y_1(t), y_2(t))$ be solution of equation (4) for $t \in [t_{k-2}, t_{k-1})$ with

$$y_i(t_{k-2}) = \lim_{t \to t_{k-2} - 0} y_i(t), \qquad i = 1, 2. \tag{7}$$

Further, using initial conditions (7) for $t \in [t_{k-1}, t_k)$ we obtain a solution of (4) as

$$y_i(t) = K_i(t_{k-1})(t - t_{k-1}) + y_i(t_{k-1}), \qquad i = 1, 2, \tag{8}$$

for $k = 2, \ldots, n$. By construction, the functions $y_1(t)$ and $y_2(t)$ defined by (8) are differentiable on

$$(a, t_1) \cup (t_1, t_2) \cup \cdots \cup (t_{n-1}, b),$$

and continuous on $[a,b]$. It is clear that the vector function $(y_1(t), y_2(t))$ is unique solution for the equation (4).

Proof of the Theorem 2. Let $(y_1(t), y_2(t))$ be the solution of equation (7). Then, for $t \in [t_k, t_{k+1})$, the following identity holds:

$$y_i'(t) - f_i\left(t, y_1(t), y_2(t)\right) = K_i(t_{k-1}) - f_i\left(t, y_1(t), y_2(t)\right).$$

We estimate the right-hand side as follows:

$$|K_i(t_{k-1}) - f_i\,(t, y_1(t), y_2(t))\,| \le |K_i(t_{k-1}) - f_i\,(t_{k-1}, y_1(t_{k-1}), y_2(t_{k-1}))\,|+$$

$$+|f_i\,(t_{k-1}, y_1(t_{k-1}), y_2(t_{k-1})) - f_i\,(t, y_1(t), y_2(t))\,|.$$

According to equation (5), there exists a constant $C > 0$ such that

$$|K_i(t_{k-1}) - f_i\big(t_{k-1}, y_1(t_{k-1}), y_2(t_{k-1})\big)| \le \frac{C}{n}$$

for sufficiently large n. By equation (8), the function $y_i(t)$ on $[t_k, t_{k+1})$ can be written as:

$$y_i(t) = K_i(t_{k-1})(t - t_{k-1}) + y_i(t_{k-1}),$$

so that
$f_i\big(t, y_1(t), y_2(t)\big) = f_i\big(t, K_1(t_{k-1})(t - t_{k-1}) + y_i(t_{k-1}), K_2(t_{k-1})(t - t_{k-1}) + y_i(t_{k-1})\big), \quad t \in [t_{k-1}, t_k)$.

Since the function f is differentiable on the domain D, there exists a constant $F > 0$ such that

$$|f_i(t, y_1(t_{k-1}), y_2(t_{k-1})) - f_i(t, y_1(t), y_2(t))| \le \frac{F}{n},$$

for all $t \in [t_{k-1}, t_k)$ and sufficiently large n.

Combining the above estimates yields:

$$\big|y_i'(t) - f_i\big(t, y_1(t), y_2(t)\big)\big| \le \frac{C+F}{n},$$

for all $t \in [t_{k-1}, t_k)$, $k = 1, \dots, n$, and sufficiently large n. This completes the proof of the theorem. In order to verify the effectiveness of the proposed methods, we consider several examples and give the absolute errors of approximate and exact solutions.

3. Summary of Algorithms

Algorithms for solving the initial conditional problem are presented as follows:

Input: Input initial data.
Step 1: Find $\big(y_{1n}(t), y_{2n}(t)\big)$ in the system of equations (6) for $t \in [t_0, t_1)$.
Step 2: Find limits

$$y_{1n}(t_1) = \lim_{t \to t_1 - 0} y_{1n}(t), \quad y_{2n}(t_1) = \lim_{t \to t_1 - 0} y_{2n}(t).$$

Step 3: According to (8), determine the functions $(y_{1n}(t), y_{2n}(t))$ for $t \in [t_{k-1}, t_k)$, $k = 2, \dots, n$, and find the limits

$$y_{1n}(t_{k-1}) = \lim_{t \to t_{k-1} - 0} y_{1n}(t), \qquad y_{2n}(t_{k-1}) = \lim_{t \to t_{k-1} - 0} y_{2n}(t).$$

Step 4: We construct an approximate solution from (8).
Step 5: We draw the graph of the approximate solution and compare it with the graph of the exact solution by placing it on one coordinate.
Step 6: Evaluate the maximum absolute error between the exact solution $y_p(t)$ and the numerical approximation $y_{pn}(t)$ and the rate of convergence:

$$L_\infty(y_p, n) = \max_{a \le t \le b}(|y_p(t) - y_{pn}(t)|).$$

$$\text{Rate of convergence} = log\big(L_\infty(y_p, n-1))/L_\infty(y_p, n)\big)/log(2), \qquad p = 1, 2.$$

Output: If the errors at the specified point are acceptable, then the for loop ends.

4. Numerical Results

To obtain an approximate solution of equation (1) from Theorem 2, it is necessary to compute $(y_{1n},\, y_{2n})$, which is given explicitly in (8). This computation relies on evaluating the increment function $K_i(t_k)$ within the framework of the seventh-order method [16]. To assess the effectiveness of the proposed methods, we present several illustrative examples and report the absolute errors between the approximate and exact solutions.

Test Problem 1. *Taking the linear stiff system* [5]

$$y_1'(t) = -y_1(t) + 95y_2(t),\; y_2'(t) = -y_1(t) - 97y_2(t),$$

with given initial conditions $y_1(0) = 1, \quad y_2(0) = 1$. *The exact solution is* $y_1(t) = \frac{95e^{-2t}-48e^{-96t}}{47}, \quad y_2(t) = \frac{48e^{-96t}-e^{-2t}}{47}$.

The results for Problem 1 are presented using the higher-order Haar wavelet collocation method (H-HWCM) proposed in [5]. This method demonstrates superior performance compared to both the standard HWCM and the nonuniform Haar wavelet methods, where for the case, when the problem is stiff ODEs, the nonuniform Haar wavelet method provides higher accuracy due to its use of variable step sizes, as discussed in [17].

The Problem 1 has been solved by present method and the results are compared with the higher-order Haar wavelet collocation method (H-HWCM) [5] (see Table 1), where present method is better than H-HWCM method. The numerical solution of the given problem shows that present method gives much more accurate results than the H-HWCM method and is also time efficient. The maximum error, experimental order of convergence, and the CPU time are given in Table 1 to show the performance and efficiency of the proposed Present method. The comparison of numerical and exact solutions along with pointwise error obtained by present method are presented in Fig. 1.

Table 1

Numerical results for Test Problem 1 obtained using the higher-order Haar wavelet collocation method (H-HWCM) and the present method, where RC denotes a rate of convergence

	H-HWCM method				Present method			
N	$L_\infty(y_1)$	$L_\infty(y_2)$	RC	CPU	$L_\infty(y_1)$	$L_\infty(y_2)$	RC	CPU
32	$5,6470E-02$	$5,6470E-02$	–	$0,11$	$7,7242E-03$	$7,7242E-03$	–	$2,52$
64	$6,6887E-03$	$6,6887E-03$	3,08	$0,29$	$1,4138E-05$	$1,4138E-05$	9,09	$2,73$
128	$5,3416E-04$	$5,3416E-04$	3,65	$0,85$	$3,2308E-08$	$3,2308E-08$	8,77	$2,93$
256	$4,2096E-05$	$4,2096E-05$	3,67	$3,48$	$1,0206E-10$	$1,0206E-10$	8,31	$3,37$
512	$2,9462E-06$	$2,9462E-06$	3,84	$13,27$	$3,5877E-13$	$3,5877E-13$	8,15	$4,23$

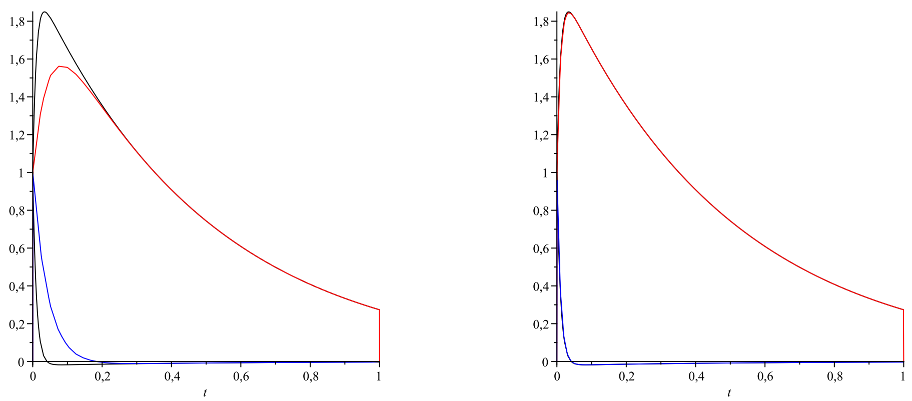

Fig. 1. Graphs of the exact solutions (black) and approximate solutions (red and blue) for Test Problem 1, with $N = 40$ and $N = 100$, respectively

Test Problem 2. *Consider the nonlinear system of differential equations with variable coefficients* [5]

$$y_1'(t) = \cos(t) + y_1^2(t) + y_2(t) - (1 + t^2 + \sin^2(t)),$$
$$y_2'(t) = 2t - (1 + t^2)\sin(t) + y_1(t)y_2(t)$$

with given initial conditions $y_1(0) = 0$ and $y_2(0) = 1$. *The exact solution is* $y_1(t) = sin(t), \quad y_2(t) = 1 + t^2$.

Table 2

Comparison of H-HWCM with Present method for Test Problem 2

N	HWCM		H-HWCM		Present method	
	$L_\infty(y_1)$	$L_\infty(y_2)$	$L_\infty(y_1)$	$L_\infty(y_2)$	$L_\infty(y_1)$	$L_\infty(y_2)$
16	1,7431E-03	2,4924E-03	1,6904E-04	1,0579E-05	2,8361E-11	4,3974E-11
32	4,5390E-04	6,4206E-04	8,0472E-05	2,5718E-06	7,3594E-14	9,4638E-14
64	1,1598E-04	1,6315E-04	3,9270E-05	6,3332E-07	3,8342E-16	5,2551E-16
128	2,9324E-05	4,1138E-05	1,9399E-05	1,5708E-07	2,4825E-18	3,6010E-18

Table 2 presents a comparison of the numerical results for Problem 2 between the proposed method and the H-HWCM method in [5]. The proposed method demonstrates higher accuracy, as indicated by the smaller absolute errors at $t_i \in [0, 2]$. A comparison of the numerical solutions obtained by the proposed method with the exact solution is also shown in Fig. 2.

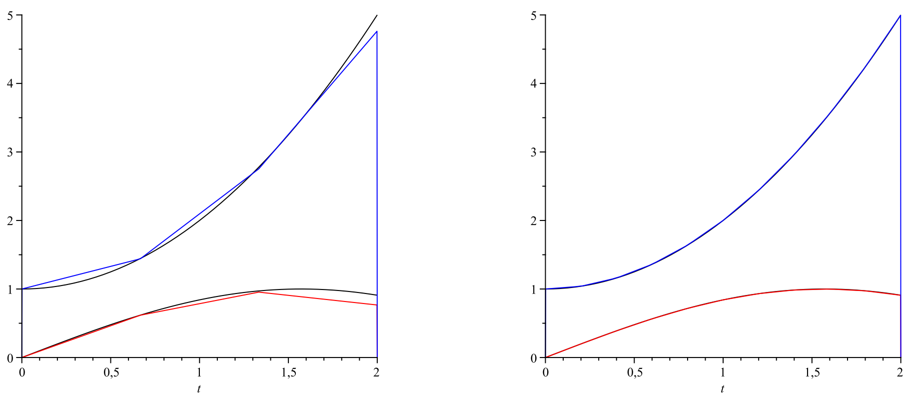

Fig. 2. The graphs of original (black) and approximate (red and blue) solutions for Test Problem 2, respectively as $N = 3$ and $N = 10$

Test Problem 3. *Source* [18]

$$y_1'(t) = 42y_2(t) - 43y_1(t),$$
$$y_2'(t) = -8y_2(t) + 7y_1(t),$$
$$y_1(0) = 8, \quad y_2(0) = 1, \quad 0 \le t \le 1.$$

Exact solution: $y_1(t) = 2e^{-t} + 6e^{-50t}, \quad y_2(t) = 2e^{-t} - e^{-50t}$.

The evaluation of maximum error, MAXE is computed as follows,

$$\text{MAXE} = \max_{0 \le t_k \le N} \left(\max_{1 \le i \le 2} E(y_i(t_k)) \right), \quad i = 1, 2,$$

where N is the total number of step, $E(y_i(t)) = |y_{ni}(t) - y_i(t)|$, where y_i is the exact solution for the problem and y_{ni} is the computed solution. The results obtained using various methods, including block backward differentiation formulas (BBDFs), a class of hybrid block backward differentiation formulas (HBBDFs), and the stiff solver ode15s as reported in [18], have been compared with those of the proposed method. The comparison is presented in Tabl. 3 and illustrated in Fig. 3.

Table 3

Numerical results for Test Problem 3

h	Method	MAXE
10^{-2}	HBBDF	2,3743E-01
	BBDF(5)	1,9713E-01
	ode15s	5,0258E-02
	Present method	6,5190E-09
10^{-4}	HBBDF	9,4970E-05
	BBDF(5)	3,9449E-04
	ode15s	3,9699E-04
	Present method	4,8760E-25

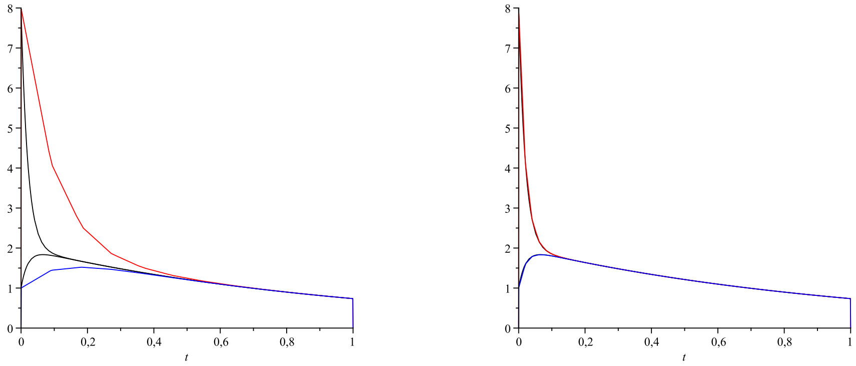

Fig. 3. The graphs of original (black) and approximate (red and blue) solutions for Test Problem 3, respectively as $N = 10$ and $N = 50$

Conclusion

A new method is presented for finding approximate solution of the system of equations. First, using the coefficients of a seventh-order Runge–Kutta method a system of differential equation with piecewise constant arguments corresponding to the considering initial value problem has been constructed. The definition of solution for ordinary differential equations with piecewise constant argument has been introduced as a piecewise-smooth function with respect to the positive integer number N. It has been proved that a unique piecewise-smooth solution is an approximate solution of the considering initial value problem for large N.

The test problems have been solved using the proposed method, and the results are compared with those obtained by the higher-order Haar wavelet collocation method (H-HWCM) [5], the block backward differentiation formulas (BBDFs), a class of hybrid block backward differentiation formulas (HBBDFs), and the stiff solver ode15s as reported in [18]. The numerical results demonstrate the effectiveness of the proposed method, and the errors between the approximate and exact solutions are evaluated to assess its accuracy.

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КОМБИНИРОВАННЫЙ МЕТОД РУНГЕ – КУТТЫ И МЕТОД КУСОЧНО-ПОСТОЯННОГО АРГУМЕНТА ДЛЯ НЕЛИНЕЙНЫХ СИСТЕМ ДИФФЕРЕНЦИАЛЬНЫХ УРАВНЕНИЙ

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Разработан подход, сочетающий метод кусочно-постоянного аргумента и метод Рунге – Кутты, для решения нелинейных систем обыкновенных дифференциальных уравнений первого порядка. С использованием коэффициентов метода Рунге – Кутты седьмого порядка строится дифференциальное уравнение с кусочно-постоянным аргументом, соответствующее рассматриваемой задаче Коши и зависящее от целого положительного числа N. Доказано, что это уравнение имеет единственное кусочно-гладкое решение, которое при больших значениях N служит приближенным решением исходной задачи Коши. Эффективность предложенного метода продемонстрирована на ряде численных примеров, показавших более высокую точность и скорость сходимости по сравнению с методом коллокации на основе вейвлетов Хаара высокого порядка и другими методами, описанными в литературе.

Ключевые слова: обыкновенное дифференциальное уравнение; кусочно-постоянный аргумент; приближенное решение; метод Рунге – Кутты; абсолютная погрешность.

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