MSC 31B30

ON THE MEAN-VALUE PROPERTY FOR POLYHARMONIC FUNCTIONS

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The mean-value property for normal derivatives of polyharmonic function on the unit sphere is obtained. The value of integral over the unit sphere of normal derivative of mth order of polyharmonic function is expressed through the values of the Laplacian's powers of this function at the origin. In particular, it is established that the integral over the unit sphere of normal derivative of degree not less then $2k-1$ of k-harmonic function is equal to zero. The values of polyharmonic function and its Laplacian's powers at the center of the unit ball are found. These values are expressed through the integral over the unit sphere of a linear combination of the normal derivatives up to $k-1$ degree for the k-harmonic function. Some illustrative examples are given.

Keywords: polyharmonic functions, mean-value property, normal derivatives on a sphere.

Introduction

In investigation of mathematical models described by the polyharmonic equation properties of polyharmonic functions are very useful to know. Let $u(x)$ be a harmonic function in the domain $\Omega \subset \mathbb{R}^n$ and $B_r(x) = \{y \in \mathbb{R}^n : |y-x| < r\}$. It is well known (see [1]) the Gauss mean-value property for harmonic functions: if $x \in \Omega$ and $\overline{B_r(x)} \subset \Omega$, then for all functions harmonic in Ω

$$u(x) = \frac{1}{|\partial B_r(x)|} \int_{\partial B_r(x)} u(y)\, ds_y. \tag{1}$$

This mean-value property has been extended by Pizzetti (see [2]) for k-harmonic functions in Ω to the form

$$\frac{1}{|\partial B_r(x)|} \int_{\partial B_r(x)} u(y)\, ds_y = \Gamma(n/2) \sum_{i=0}^{k-1} \frac{r^{2i} \Delta^i u(x_0)}{4^i i! \Gamma(i+n/2)},$$

where $\Gamma(\alpha)$ is the Euler's gamma function. This property can be easily written for a k-harmonic function $u \in C^{k-1}(\bar{S})$ in the unit ball $S \subset \mathbb{R}^n$ in the form

$$\frac{1}{\omega_n} \int_{\partial S} u(x)\, ds_x = \sum_{i=0}^{k-1} \frac{\Delta^i u(0)}{(2,2)_i (n,2)_i}, \tag{2}$$

where ω_n is the surface area of the unit sphere ∂S, and $(a,b)_k = a(a+b)\cdots(a+b(k-1))$ is the generalized Pochhammer symbol with $(a,b)_0 = 1$. For example, $(2,2)_i = (2i)!!$. The similar formula was proved in [3, Theorem 7] for calculating the integral of homogeneous polynomial $Q_m(x)$ on the unit sphere

$$\int_{|x|=1} Q_m(x)\, ds_x = \begin{cases} 0, & m \in 2\mathbb{N}-1 \\ \dfrac{\Delta^{m/2} Q_m(x)}{m!!\, n \cdots (n+m-2)}\, \omega_n, & m \in 2\mathbb{N} \end{cases}.$$

Consider the operator Λ defined by the equality

$$\Lambda = \sum_{i=1}^{n} x_i \frac{\partial}{\partial x_i}. \tag{3}$$

This operator plays an important role in our investigation because in the paper [4] it was proved that the following equality is fulfilled on ∂S

$$\frac{\partial^k u}{\partial \nu^k} = \Lambda^{[k]} u, \tag{4}$$

where ν is the outer normal to ∂S, and $t^{[k]} = t(t-1)\dots(t-k+1)$ is a factorial power of t. Besides, it is known (see, for example, [5]) that if u is a harmonic function, then function $P(\Lambda)u$ is also a harmonic one, where $P(\lambda)$ is a polynomial.

1. The mean-value property for normal derivatives

We are going to extend formula (2) to the normal derivatives of the function $u(x)$. Let us denote $\mathbb{N}_0 \equiv \mathbb{N} \cup \{0\}$.

Theorem 1. *For all $m \in \mathbb{N}_0$ and for any polyharmonic function in the unit ball $u \in C^m(\bar{S})$ the following equality holds*

$$\frac{1}{\omega_n} \int_{\partial S} \frac{\partial^m u}{\partial \nu^m} \, ds_x = \sum_{k=0}^{\infty} \frac{(2k)^{[m]}}{(2,2)_k (n,2)_k} \Delta^k u(0), \tag{5}$$

where ν is the unit outer normal to ∂S.

Proof. In [6, Theorem 4] it is proved that for any polyharmonic in S function $u(x)$ the following Almansi representation takes place

$$u(x) = v_0(x) + \sum_{k=1}^{\infty} \frac{1}{4^k} \frac{|x|^{2k}}{k!} \int_0^1 \frac{(1-\alpha)^{k-1}}{(k-1)!} \alpha^{n/2-1} v_k(\alpha x) \, d\alpha, \tag{6}$$

where harmonic in S functions $v_0(x), \dots, v_k(x), \dots$ are given by the formula

$$v_k(x) = \Delta^k u(x) + \sum_{s=1}^{\infty} \frac{(-1)^s}{4^s} \frac{|x|^{2s}}{s!} \int_0^1 \frac{(1-\alpha)^{s-1} \alpha^{s-1}}{(s-1)!} \alpha^{n/2-1} \Delta^{k+s} u(\alpha x) \, d\alpha. \tag{7}$$

The upper limit of sum above is equal to infinity but since the function $u(x)$ is a polyharmonic in S then summation is finite and exists k_0 such that $v_k(x) = 0$ for all $k \geq k_0$. It is not hard to see that

$$\Lambda\big(|x|^{2k} u\big) = |x|^{2k} (2k + \Lambda) u$$

and therefore

$$\Lambda^{[2]}\big(|x|^{2k} u\big) = (\Lambda - 1)\big(|x|^{2k}(2k + \Lambda)u\big) = |x|^{2k}(2k - 1 + \Lambda)(2k + \Lambda)u,$$

whence

$$\Lambda^{[m]}\big(|x|^{2k} u\big) = |x|^{2k}(2k - m + 1 + \Lambda) \cdots (2k - 1 + \Lambda)(2k + \Lambda)u = (2k)^{[m]} u + Q_m(\Lambda) u,$$

where $Q_m(\lambda)$ is a certain polynomial such that $Q_m(0) = 0$. Therefore in S we have

$$\Lambda^{[m]}u(x) = \Lambda^{[m]}v_0(x) + \sum_{k=1}^{\infty} \frac{(2k)^{[m]}|x|^{2k}}{4^k k!} \int_0^1 \frac{(1-\alpha)^{k-1}}{(k-1)!} \alpha^{n/2-1} v_k(\alpha x)\, d\alpha + \\ + \sum_{k=1}^{\infty} \frac{|x|^{2k}}{4^k k!} \int_0^1 \frac{(1-\alpha)^{k-1}}{(k-1)!} \alpha^{n/2-1} Q_m(\Lambda) v_k(\alpha x)\, d\alpha.$$

Using the mean-value property for harmonic functions and harmonicity of Λv_k we can obtain the equality $\int_{\partial S} Q_m(\Lambda) v_k(\alpha x)\, ds_x = 0$. Therefore using (7) we have

$$\frac{1}{\omega_n} \int_{\partial S} \Lambda^{[m]} u(x)\, ds_x = \sum_{k=1}^{\infty} \frac{(2k)^{[m]}}{4^k k!(k-1)!} \int_0^1 (1-\alpha)^{k-1} \alpha^{n/2-1}\, d\alpha\, v_k(0) = \\ = \sum_{k=1}^{\infty} \frac{(2k)^{[m]} v_k(0)}{4^k k!(k-1)!} \frac{\Gamma(k)\Gamma(n/2)}{\Gamma(k+n/2)} = \sum_{k=1}^{\infty} \frac{(2k)^{[m]} v_k(0)}{(2,2)_k (n,2)_k} = \sum_{k=0}^{\infty} \frac{(2k)^{[m]} \Delta^k u(0)}{(2,2)_k (n,2)_k}.$$

Hence, by virtue of (4), we obtain the theorem's statement for $m > 0$. If $m = 0$, then by equality (2) the formula (5) is true in this case also.

□

Example 1. Let function $u(x)$ be a harmonic in S and $u \in C^{\infty}(\bar{S})$, then from Theorem 1 follows that

$$\int_{\partial S} \frac{\partial^m u}{\partial \nu^m}\, ds_x = 0, \quad m \geq 1.$$

For a biharmonic in S function $u \in C^{\infty}(\bar{S})$ from Theorem 1 follows that

$$\int_{\partial S} \frac{\partial^m u}{\partial \nu^m}\, ds_x = \omega_n \frac{2^{[m]}}{2n} \Delta u(0) = 0, \quad m \geq 3$$

since $2^{[m]} = 0$ at $m \geq 3$ (see example 3). In general case, if the function $u(x)$ is a k-harmonic in S and $u \in C^{\infty}(\bar{S})$, them from Theorem 1 it follows that

$$\int_{\partial S} \frac{\partial^m u}{\partial \nu^m}\, ds_x = \omega_n \sum_{i=0}^{k-1} \frac{(2i)^{[m]} \Delta^i u(0)}{(2,2)_i (n,2)_i} = 0, \quad m \geq 2k-1$$

because of equality $(2k-2)^{[m]} = 0$ provided that $m \geq 2k-1$.

2. The value of polyharmonic function at the unit ball center

The following statement is true.

Theorem 2. *For any polyharmonic in the unit ball $S \subset \mathbb{R}^n$ function $u \in C^{k-1}(\bar{S})$ the equality*

$$u(0) = \frac{1}{\omega_n} \int_{\partial S} \left(h_k^0 u + h_k^1 \frac{\partial u}{\partial \nu} + \cdots + h_k^{k-1} \frac{\partial^{k-1} u}{\partial \nu^{k-1}} \right) ds_x \tag{8}$$

holds, where h_k^s are found from the equality

$$h_k^s = \frac{(-1)^{k-1}}{s!(k-1)!} \left(\frac{1}{t} (\sqrt{t} - 1)^s \right)^{(k-1)}_{|t=1}, \tag{9}$$

satisfy to the recurrence relation

$$h_{k+1}^s = \left(1 - \frac{s}{2k}\right) h_k^s + \frac{1}{2k} h_k^{s-1}, \tag{10}$$

and are coefficients of the polynomial

$$H_{k-1}(\lambda) = \frac{(-1)^{k-1}}{(2k-2)!!}(\lambda-2)\cdots(\lambda-2k+2) \qquad (11)$$

expanded in the terms of factorial powers $\lambda^{[s]}$

$$H_{k-1}(\lambda) = h_k^{k-1}\lambda^{[k-1]} + h_k^{k-2}\lambda^{[k-2]} + \cdots + h_k^1\lambda^{[1]} + h_k^0. \qquad (12)$$

The original proof of this Theorem is omitted because it requires some additional investigations and moreover this Theorem is a special case of more general Theorem 4.

It is necessary to note that recurrence relation similar to (10) $a_{k+1}^s = (k-2s+1)a_k^s + \frac{1}{2}a_k^{s-1}$ was used in [7], where special polynomials were constructed. Regularization of integral equations was considered in [8, 9]. Recurrence relation of the form (10) determines some arithmetical triangle similar to Pascal, Euler and Stirling triangles, but its elements are rational fractions. Calculating h_k^s by the formula (10) the triangle H can be written in the form

$$H = \begin{array}{ccccccccc} & & & & 1 & & & & \\ & & & 1 & & -\frac{1}{2} & & & \\ & & 1 & & -\frac{5}{8} & & \frac{1}{8} & & \\ & 1 & & -\frac{11}{16} & & \frac{3}{16} & & -\frac{1}{48} & \\ 1 & & -\frac{93}{128} & & \frac{29}{128} & & -\frac{7}{192} & & \frac{1}{384} \\ & & & & \cdots & & & & \\ & \cdots & & & h_{k+1}^s = (1-s/(2k))h_k^s - 1/(2k)h_k^{s-1} & & & \cdots & \end{array} \qquad (13)$$

Remark 1. Formula (8) according to (12) and (4) can be represented in the form

$$u(0) = \frac{1}{\omega_n}\int_{\partial S} H_{k-1}(\Lambda)u(x)\,ds_x.$$

Example 2. For a 4-harmonic function $u \in C^3(\bar{S})$, according to 4th row of the triangle H from (13), the following equality holds

$$u(0) = \frac{1}{\omega_n}\int_{\partial S}\Big(u - \frac{11}{16}\frac{\partial u}{\partial \nu} + \frac{3}{16}\frac{\partial^2 u}{\partial \nu^2} - \frac{1}{48}\frac{\partial^3 u}{\partial \nu^3}\Big)\,ds_x.$$

Consider polynomial

$$H_{k-1}^{(m)}(\lambda) = \lambda(\lambda-2)\cdots(\lambda-2m+2)(\lambda-2m-2)\cdots(\lambda-2k+2). \qquad (14)$$

It is obvious, that $H_{k-1}(\lambda) = H_{k-1}^{(0)}(\lambda)/H_{k-1}^{(0)}(0)$ and $H_{k-1}^{(m)}(2m) \neq 0$.

Lemma 1. *Let*

$$u(x) = u_0(x) + \cdots + |x|^{2k-2}u_{k-1}(x)$$

be the Almansi representation of a k-harmonic in S function $u(x)$ *and such that* $u \in C^{k-1}(\bar{S})$, *then for* $m \in \mathbb{N}_0$ *and* $m < k$ *the equality*

$$u_m(0) = \frac{1}{\omega_n H_{k-1}^{(m)}(2m)}\int_{\partial S} H_{k-1}^{(m)}(\Lambda)u(x)\,ds_x \qquad (15)$$

holds true.

Proof. Let $\lambda \in \mathbb{R}$, $i \in \mathbb{N}_0$, $i < k$ and $v(x)$ be a harmonic in S function. It is not hard to see that in S the equality

$$(\Lambda - \lambda)\Big(|x|^{2i} v(x)\Big) = |x|^{2i}\big((2i - \lambda)v(x) + \Lambda v(x)\big)$$

holds true and therefore

$$H_{k-1}^{(m)}(\Lambda)\Big(|x|^{2i} v(x)\Big) = |x|^{2i}\Big(H_{k-1}^{(m)}(2i)v(x) + Q_{k-1}(\Lambda)v(x)\Big),$$

where $Q_{k-1}(\lambda)$ is a certain polynomial of degree $(k-1)$ depending on $H_{k-1}^{(m)}$ and such that $Q_{k-1}(0) = 0$. Function $Q_{k-1}(\Lambda)v$ is also a harmonic in S function. Let S_r be a sphere of the radius r with a center at the origin of coordinates. For all $r \in (0,1)$ we have $Q_{k-1}(\Lambda)v \in C(\bar{S}_r)$. Then

$$\int_{\partial S_r} Q_{k-1}(\Lambda)v(x)\, ds_x = Q_{k-1}(0)\int_{\partial S_r} v(x)\, ds_x = 0.$$

Therefore, if $i \neq m$, then $H_{k-1}^{(m)}(2i) = 0$ and then

$$\int_{\partial S_r} H_{k-1}^{(m)}(\Lambda)\Big(|x|^{2i} v(x)\Big)\, ds_x = H_{k-1}^{(m)}(2i)\int_{\partial S_r} v(x)\, ds_x + \int_{\partial S_r} Q_{k-1}(\Lambda)v(x)\, ds_x = 0.$$

If $i = m$ then similarly to the above

$$\frac{1}{\omega_n^r}\int_{\partial S_r} H_{k-1}^{(m)}(\Lambda)\Big(|x|^{2m} v(x)\Big)\, ds_x = H_{k-1}^{(m)}(2m)\frac{1}{\omega_n^r}\int_{\partial S_r} v(x)\, ds_x = H_{k-1}^{(m)}(2m)v(0),$$

where ω_n^r is the surface area of the sphere ∂S_r. Therefore for the function $u(x)$ the equality

$$\frac{1}{\omega_n^r}\int_{\partial S_r} H_{k-1}^{(m)}(\Lambda)u(x)\, ds_x = \sum_{i=0}^{k-1}\frac{1}{\omega_n^r}\int_{\partial S_r} H_{k-1}^{(m)}(\Lambda)\Big(|x|^{2i} u_i(x)\Big)\, ds_x = H_{k-1}^{(m)}(2m)u_m(0). \qquad (16)$$

holds. Since $u \in C^{k-1}(\bar{S})$, then dividing this equality on $H_{k-1}^{(m)}(2m) \neq 0$ and taking the limit as $r \to 1$ we obtain the lemma's statement (15).

□

Theorem 3. *For any k-harmonic in the unit ball S function $u \in C^k(\bar{S})$ the equality*

$$\int_{\partial S} H_k^{(k)}(\Lambda)u(x)\, ds_x = 0,$$

holds, where $H_k^{(k)}(\lambda) = \lambda(\lambda - 2)\cdots(\lambda - 2k + 2)$.

Proof. It is not hard to see that $\forall\, i < m$, $H_k^{(k)}(2i) = 0$. Therefore using the equality (16) from Lemma 1 at $r \in (0,1)$ we have

$$\frac{1}{\omega_n^r}\int_{\partial S_r} H_k^{(k)}(\Lambda)u(x)\, ds_x = \sum_{i=0}^{k-1}\frac{1}{\omega_n^r}\int_{\partial S_r} H_k^{(k)}(\Lambda)\Big(|x|^{2i} u_i(x)\Big)\, ds_x = 0.$$

Taking the limit for $r \to 1$ we obtain the desired equality.

□

Some generalization of the well known property of the harmonic functions $\int_{\partial S} \frac{\partial u}{\partial \nu}\, ds_x = 0$ on the polyharmonic functions is the following assertion.

Sequence 1. *If the numbers a_i are found from the equality*

$$H_k^{(k)}(\lambda) = \lambda^{[k]} + a_{k-1}\lambda^{[k-1]} + \cdots + a_1\lambda^{[1]} + a_0,$$

then for any polyharmonic in the unit ball S function $u \in C^k(\bar{S})$ the equality

$$\int_{\partial S} \Big(a_0 u + a_1 \frac{\partial u}{\partial \nu} + \cdots + a_k \frac{\partial^k u}{\partial \nu^k} \Big)\, ds_x = 0$$

holds.

To prove this corollary it is sufficient to remember (4) and to take advantage of Theorem 3. Theorem 2 can be generalized in the following way.

Theorem 4. *For any polyharmonic in the unit ball S function $u \in C^{k-1}(\bar{S})$ the equality*

$$\Delta^m u(0) = \frac{1}{\omega_n} \frac{(2,2)_m (n,2)_m}{H_{k-1}^{(m)}(2m)} \int_{\partial S} H_{k-1}^{(m)}(\Lambda) u(x)\, ds_x, \tag{17}$$

holds, where the polynomial $H_{k-1}^{(m)}(\lambda)$ is defined in (14) *and $m = 0, \ldots, k-1$.*

Proof. Let

$$u(x) = u_0(x) + \cdots + |x|^{2k-2} u_{k-1}(x) \tag{18}$$

be the Almansi representation of a k-harmonic in S function $u(x)$ and such that $u \in C^{k-1}(\bar{S})$, then for $m \in \mathbb{N}_0$ and $m < k$ the equality (15) holds. Besides, if v is a harmonic in S function, then (see [4])

$$\Delta\Big(|x|^{2m} v(x)\Big) = |x|^{2m-2} 2m(2m + n - 2 + 2\Lambda) v(x).$$

Therefore for $i < m$ we have

$$\Delta^i\Big(|x|^{2m} v(x)\Big) = |x|^{2m-2i} \prod_{j=m-i+1}^{m} 2j(2j + n - 2 + 2\Lambda) v(x)$$

and hence $\Delta^i\Big(|x|^{2m} v(x)\Big)_{|x=0} = 0$. If $i = m$, then we have

$$\Delta^m\Big(|x|^{2m} v(x)\Big) = \prod_{j=1}^{m} 2j(2j + n - 2 + 2\Lambda) v(x) = \prod_{j=1}^{m} 2j(2j + n - 2) v(x) + P_k(\Lambda) v(x) =$$
$$= 2m!!\, n \cdots (n + 2m - 2) v(x) + P_k(\Lambda) v(x) = (2,2)_m (n,2)_m v(x) + P_k(\Lambda) v(x), \tag{19}$$

where $P_k(\lambda)$ is a certain polynomial of kth power and such that $P_k(0) = 0$. Therefore we obtain

$$\Delta^m\Big(|x|^{2m} v(x)\Big)_{|x=0} = (2,2)_m (n,2)_m v(0).$$

From (18) it follows that for $i > m$

$$\Delta^i\Big(|x|^{2m} v(x)\Big) = (2,2)_m (n,2)_m \Delta^{i-m} v(x) + \Delta^{i-m} P_k(\Lambda) v(x) = 0.$$

Therefore applying the operator Δ^m for $m \in \mathbb{N}_0$ and $m < k$ to the equality (18), then assuming $x = 0$ and using (15) we obtain

$$\Delta^m u(x)_{|x=0} = \sum_{i=0}^{k-1} \Delta^m \Big(|x|^{2i} u_i(x)\Big)_{|x=0} = \frac{1}{\omega_n} \frac{(2,2)_m (n,2)_m}{H_{k-1}^{(m)}(2m)} \int_{\partial S} H_{k-1}^{(m)}(\Lambda) u(x)\, ds_x.$$

Formula (17) is proved.

□

Remark 2. It is not hard to see that if the function $u \in C^k(\bar{S})$ is a $(k+1)$-harmonic in the unit ball, then for numbers a_i from Sequence 1, according to Theorem 4, the following equality holds

$$\Delta^k u(0) = \int_{\partial S} \Big(a_0 u + a_1 \frac{\partial u}{\partial \nu} + \cdots + a_k \frac{\partial^k u}{\partial \nu^k}\Big)\, ds_x.$$

Example 3. Let the function $u(x)$ be a 3-harmonic one in S and $u \in C^2(\bar{S})$. It is easy to see that

$$H_2^{(2)}(\lambda) = \lambda(\lambda - 2) = -\lambda^{[1]} + \lambda^{[2]},$$

$H_2^{(2)}(4) = 8$, $(2,2)_2 = 8$, $(n,2)_2 = n(n+2)$ and therefore the following equality holds

$$\Delta^2 u(0) = \frac{n(n+2)}{\omega_n} \int_{\partial S} \Big(-\frac{\partial u}{\partial \nu} + \frac{\partial^2 u}{\partial \nu^2}\Big)\, ds_x.$$

If the function $u(x)$ is a biharmonic one in S, then according to Sequence 1 we obtain

$$\int_{\partial S} \Big(-\frac{\partial u}{\partial \nu} + \frac{\partial^2 u}{\partial \nu^2}\Big)\, ds_x = 0 \Rightarrow \int_{\partial S} \frac{\partial u}{\partial \nu}\, ds_x = \int_{\partial S} \frac{\partial^2 u}{\partial \nu^2}\, ds_x.$$

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УДК 517.575

О СВОЙСТВЕ СРЕДНЕГО ДЛЯ ПОЛИГАРМОНИЧЕСКИХ ФУНКЦИЙ В ШАРЕ

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Получено свойство среднего для нормальных производных от полигармонической функции по единичной сфере. Значение интеграла от нормальных производных по единичной сфере от полигармонической функции выражается через значения степеней лапласианов от этой функции в начале координат. В частности, установлено, что интеграл по единичной сфере от нормальных производных k-гармонической функции порядка не меньше $2k-1$ равен нулю. Найдены значения полигармонической функции и лапласианов от нее в центре единичного шара. Это значение выражается через интеграл по единичной сфере от линейной комбинации нормальных производных до $k-1$ порядка для k-гармонической функции. Приведены иллюстративные примеры.

Ключевые слова: полигармонические функции, свойство среднего, нормальные производные на сфере.

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Поступила в редакцию 29 апреля 2013 г.