

ON GLOBAL IN TIME SOLUTIONS FOR DIFFERENTIAL-ALGEBRAIC EQUATIONS

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We investigate differential-algebraic equations arising in mathematical models that describe some radio-technical devices. A class of differential-algebraic equations is described, for which necessary and sufficient conditions for global in time existence of solutions are proved. As well as in many papers where sufficient conditions for such equations are obtained, we reduce them to ordinary differential equations and then apply the necessary and sufficient conditions for the latter. We deal with the systems whose matrix pencil is regular and (for simplicity) the characteristic polynomial satisfies the rank-degree condition. We also require some additional conditions that allow us to reduce the differential-algebraic system to ordinary differential one.

Keywords: differential-algebraic equation; global existence of solution; proper function; complete Riemannian metric.

Introduction

In this paper we deal with the differential-algebraic equations of the form

$$\tilde{L}\dot{x}(t) = \tilde{M}x(t) + \tilde{f}(t, x(t)) \quad (1)$$

in \mathbb{R}^n where \tilde{L} and \tilde{M} are $n \times n$ constant matrices (\tilde{L} is degenerate and \tilde{M} is non-degenerate) and $\tilde{f}(t, x)$ is a vector field on \mathbb{R}^n .

Such equations arise in mathematical models describing some radio-technical devices (see [1–3]). In [3–5] (see also the bibliography therein) some sufficient conditions for global in time existence of solutions of differential-algebraic systems are obtained. In many cases the idea of the proof is to reduce a differential-algebraic equation to an ordinary differential equation by some artificial machinery and then to apply criteria for existence of global in time solutions for the latter, reformulated in terms of the system under consideration.

The main aim of this paper is to obtain necessary and sufficient conditions for existence of global in time solutions for some type of differential-algebraic equations. We use similar idea of reducing them to ordinary differential equations but after that we apply the necessary and sufficient results obtained in [6–8].

For the sake of simplicity of presentation of our methods, unlike [3–5], we do not use spectral projectors, etc., but first we transform the differential-algebraic equation to its canonical form and then formulate the existence theorems in the terms of the latter. Also for the sake of simplicity, we deal only with the systems with regular matrix pencil whose characteristic polynomial satisfies the rank-degree condition (Chistyakov's condition). We prove the necessary and sufficient conditions for a subclass of the above-mentioned systems satisfying some conditions similar to those in [5].

The case where the rank-degree condition is not assumed, can be investigated as well, but it requires additional difficulties in its presentation and we suppose to consider it in forthcoming publications.

1. Preliminaries

Consider the ordinary differential equation

$$\dot{x}(t) = X(t, x(t)) \tag{2}$$

with smooth right-hand side on a smooth finite-dimensional manifold M . Introduce the enlarged phase space $M^+ = \mathbb{R} \times M$ and the vector field X^+ on it that at the point $(t, x) \in M^+$ is given by the formula $X^+_{(t,x)} = (1, X(t, x))$.

A smooth function $f : M^+ \rightarrow \mathbb{R}$ is called proper if its preimage of every compact set in \mathbb{R} is compact in M^+ . Note that in a Euclidean space, a function f is proper if and only if $f(x) \rightarrow \infty$ as $x \rightarrow \infty$.

The derivative of a function f in the direction of the vector field X^+ is denoted by X^+f .

Recall that a Riemannian metric is given on a manifold N if in every tangent space T_mN an inner product $\langle \cdot, \cdot \rangle_m$ is given such that the family of inner products is smooth in m . The norm of a vector $Y \in T_mN$ is defined by usual formula $\|Y\| = \sqrt{\langle Y, Y \rangle_m}$. The Riemannian metric determines the Riemannian distance $\rho(m_0, m_1)$ as the infimum of length of curves connecting m_0 and m_1 in N . With respect to this distance, the Riemannian manifold turns to a metric space. The Riemannian metric is called complete, if the above-mentioned metric space is complete.

In two theorems below the role of manifold N is played by M^+ introduced above.

Theorem 1. [6–8] *All solutions of (2) exist on the entire real line $t \in (-\infty, +\infty)$ if and only if on M^+ there exists a complete Riemannian metric such that the norm $\|X^+\|$ of vector field X^+ is uniformly bounded on M^+ .*

Theorem 2. [7, 8] *All solutions of (2) exist on the entire real line $t \in (-\infty, +\infty)$ if and only if there exists a smooth proper function $\varphi : M^+ \rightarrow \mathbb{R}$ such that the absolute value $|X^+\varphi|$ of the derivative of φ in the direction of X^+ is uniformly bounded on M^+ .*

We need also some facts from the theory of matrices. Detailed explanation of this material can be found, e.g., in [9, 10].

Definition 1. *Let two $n \times n$ constant matrices A and B be given. The expression $\lambda A + B$ where λ is a real or complex valued parameter, is called the matrix pencil. The polynomial $\det(\lambda A + B)$ (with respect to λ) is called the characteristic polynomial of the pencil. If $\det(\lambda A + B)$ is not identical zero, the pencil is called regular.*

Theorem 3. *Let the matrix pencil $\lambda A + B$ be regular. Then there exist non-degenerate matrices P and Q such that*

$$P(\lambda A + B)Q = \lambda \begin{pmatrix} E_d & 0 \\ 0 & N \end{pmatrix} + \begin{pmatrix} J & 0 \\ 0 & E_{n-d} \end{pmatrix}, \tag{3}$$

where E_d and E_{n-d} are unit matrices of the corresponding dimensions, N is an upper triangle matrix consisting of Jordan boxes with zeros on diagonal and J is a certain $d \times d$ block.

Definition 2. *If the characteristic polynomial satisfies the equality*

$$\text{rank}(A) = \text{deg}(\det(\lambda A + B)), \quad (4)$$

we say that the polynomial satisfies the condition rank-degree (Chistyakov's condition).

Theorem 4. *If the characteristic polynomial satisfies the rank-degree condition, assertion of Theorem 3 holds true and formula (3) takes the form*

$$P(\lambda A + B)Q = \lambda \begin{pmatrix} E_d & 0 \\ 0 & 0 \end{pmatrix} + \begin{pmatrix} J & 0 \\ 0 & E_{n-d} \end{pmatrix}. \quad (5)$$

2. The Main Results

Consider equation (1) such that the pencil $\lambda A + B$ is regular and its characteristic polynomial satisfies the rank-degree condition. Then applying the matrices P and Q from Theorem 3, by Theorem 4 we reduce (1) to the form

$$Ly(t) = My(t) + f(t, y(t)) \quad (6)$$

where $L = P\tilde{L}Q$, $M = P\tilde{M}Q$, $y(t) = Q^{-1}x(t)$ and $f(t, y(t)) = P\tilde{f}(t, Q^{-1}x(t))$.

The block structure of the matrices in (5) determines the decomposition of \mathbb{R}^n into the direct sum $\mathbb{R}^n = \mathbb{R}^d \oplus \mathbb{R}^{n-d}$. Using this decomposition, let us also decompose every vector $y \in \mathbb{R}^n$ into the pair $y = y^{(1)} + y^{(2)}$ where $y^{(1)} \in \mathbb{R}^d$ and $y^{(2)} \in \mathbb{R}^{n-d}$, and vector $f(t, y)$ into $f(t, y) = f^{(1)}(t, y^{(1)} + y^{(2)}) + f^{(2)}(t, y^{(1)} + y^{(2)})$ where $f^{(1)}(t, y^{(1)} + y^{(2)}) \in \mathbb{R}^d$ and $f^{(2)}(t, y^{(1)} + y^{(2)}) \in \mathbb{R}^{n-d}$. Taking into account formula (5), after that we can rewrite (6) in the form

$$\begin{pmatrix} E_d & 0 \\ 0 & 0 \end{pmatrix} \begin{pmatrix} \dot{y}^{(1)}(t) \\ \dot{y}^{(2)}(t) \end{pmatrix} = \begin{pmatrix} J & 0 \\ 0 & E_{n-d} \end{pmatrix} \begin{pmatrix} y^{(1)}(t) \\ y^{(2)}(t) \end{pmatrix} + \begin{pmatrix} f^{(1)}(t, y^{(1)}(t) + y^{(2)}(t)) \\ f^{(2)}(t, y^{(1)}(t) + y^{(2)}(t)) \end{pmatrix}. \quad (7)$$

Equation (7) evidently can be presented as the system of differential and functional equations

$$\begin{cases} \dot{y}^{(1)}(t) = Jy^{(1)}(t) + f^{(1)}(t, y^{(1)}(t) + y^{(2)}(t)), \\ y^{(2)}(t) + f^{(2)}(t, y^{(1)}(t) + y^{(2)}(t)) = 0 \end{cases}. \quad (8)$$

Condition 1. *We suppose that for every point $(t, y^{(1)}) \in \mathbb{R} \times \mathbb{R}^d$ there exists a unique point $y^{(2)} = \Phi(t, y^{(1)}) \in \mathbb{R}^{n-d}$ that is smooth jointly in $(t, y^{(1)})$ and such that $-f^{(2)}(t, y^{(1)} + \Phi(t, y^{(1)})) = \Phi(t, y^{(1)}) = y^{(2)}$.*

Remark 1. Under Condition 1 one can easily see that $y^{(2)} = \Phi(t, y^{(1)}) \in \mathbb{R}^{n-d}$ is a unique fixed point of the operator $-f^{(2)}(t, y^{(1)} + (\cdot)) : \mathbb{R}^{n-d} \rightarrow \mathbb{R}^{n-d}$. This fixed point exists, e.g., if $f^{(2)}(t, y^{(1)} + (\cdot))$ is Lipschitz continuous, i.e., if there exists $k \in (0, 1)$ such that for every pair of points $z_1, z_2 \in \mathbb{R}^{n-d}$ and every $(t, y^{(1)}) \in \mathbb{R} \times \mathbb{R}^d$ the inequality $\|f^{(2)}(t, y^{(1)} + z_1) - f^{(2)}(t, y^{(1)} + z_2)\| < k\|z_1 - z_2\|$ holds. Note that k may be a smooth function depending on $(t, y^{(1)})$. The fact that $y^{(2)} = \Phi(t, y^{(1)}) \in \mathbb{R}^{n-d}$ is smooth jointly in $(t, y^{(1)})$ can be derived from the implicit function theorem. A condition of this sort in the language of functions $y^{(1)}(t)$ and $y^{(2)}(t)$ is used in [5].

If Condition 1 is satisfied, system (8) is equivalent to the equation

$$\dot{y}^{(1)}(t) = Jy^{(1)}(t) + f^{(1)}(t, y^{(1)}(t) + \Phi(t, y^{(1)}(t))) \quad (9)$$

in \mathbb{R}^d . Since the right-hand side of (9) is smooth, for every initial condition $y^{(1)}(0) = y_0 \in \mathbb{R}^d$ the equation has a unique (local) solution.

Condition 2. *The function $f(t, x)$ is given as follows: for every $t \in \mathbb{R}$ and every $y = y^{(1)} + y^{(2)} \in \mathbb{R}^n = \mathbb{R}^d \times \mathbb{R}^{n-d}$ the equality $f^{(2)}(t, y^{(1)} + y^{(2)}) = -y^{(2)}$ holds.*

Under Condition 2 the situation is trivial. Nevertheless we investigate this case since Condition 2 is a very special example of Condition 1 and it allows us to obtain another sort of necessary and sufficient condition for global in time existence of solution. Under Condition 2 in system (8) we set $y^{(2)}(t) = \text{const}$ and so reduce (8) to the system

$$\begin{cases} \dot{y}^{(1)}(t) = Jy^{(1)}(t) + f^{(1)}(t, y^{(1)}(t) + y^{(2)}), \\ y^{(2)} = \text{const}. \end{cases} \quad (10)$$

in \mathbb{R}^n . Since the right-hand side of (10) is smooth, for every initial condition $y(0) = y_0 = y_0^{(1)} + y_0^{(2)}$ in \mathbb{R}^n there exists a (local) solution of (10).

In obtaining the necessary and sufficient conditions for global in time existence of solutions of (1), first we consider the case where Condition 1 is satisfied. Denote by $Y^+(t, y^{(1)})$ the vector field $(1, Jy^{(1)} + f^{(1)}(t, y^{(1)} + \Phi(t, y^{(1)})))$ in \mathbb{R}^{d+1} .

Theorem 5. *Let the pencil $\lambda\tilde{L} + \tilde{M}$ be regular, the characteristic polynomial satisfy the rank-degree condition and Condition 1 be fulfilled. Then all solutions of (1) exist on $(-\infty, +\infty)$ if and only if there exists a complete Riemannian metric on \mathbb{R}^{d+1} such that the norm of vector field Y^+ is uniformly bounded on \mathbb{R}^{d+1} .*

Theorem 6. *Let the pencil $\lambda\tilde{L} + \tilde{M}$ be regular, the characteristic polynomial satisfy the rang-degree condition and Condition 1 be fulfilled. Then all solutions of (1) exist on $(-\infty, +\infty)$ if and only if on \mathbb{R}^{d+1} there exists a smooth proper function $\varphi : \mathbb{R}^{d+1} \rightarrow \mathbb{R}$ such that the absolute value $|Y^+\varphi|$ of derivative of φ in the direction of Y^+ is uniformly bounded on \mathbb{R}^{d+1} .*

Indeed, Theorem 5 (Theorem 6) is a reformulation of Theorem 1 (Theorem 2, respectively) for equation (9) on \mathbb{R}^d .

Now let us turn to the case, in which Condition 2 is satisfied. Here we introduce the vector field $\bar{Y}^+(t, y^{(1)}, y^{(2)}) = (1, Jy^{(1)}(t) + f^{(1)}(t, y^{(1)}(t) + y^{(2)}), 0)$ on \mathbb{R}^{n+1} .

Theorem 7. *Let the pencil $\lambda\tilde{L} + \tilde{M}$ be regular, the characteristic polynomial satisfy the rank-degree condition and Condition 2 be fulfilled. Then all solutions of (1) exist on $(-\infty, +\infty)$ if and only if there exists a complete Riemannian metric on \mathbb{R}^{n+1} such that the norm of vector field \bar{Y}^+ is uniformly bounded on \mathbb{R}^{n+1} .*

Theorem 8. *Let the pencil $\lambda\tilde{L} + \tilde{M}$ be regular, the characteristic polynomial satisfy the rang-degree condition and Condition 2 be fulfilled. Then all solutions of (1) exist on $(-\infty, +\infty)$ if and only if on \mathbb{R}^{n+1} there exists a smooth proper function $\varphi : \mathbb{R}^{n+1} \rightarrow \mathbb{R}$ such that the absolute value $|\bar{Y}^+\varphi|$ of derivative of φ in the direction of \bar{Y}^+ is uniformly bounded on \mathbb{R}^{n+1} .*

Indeed, Theorem 7 (Theorem 8) is a reformulation of Theorem 1 (Theorem 2, respectively) for system (10) on \mathbb{R}^n .

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О ГЛОБАЛЬНЫХ ПО ВРЕМЕНИ РЕШЕНИЯХ ДИФФЕРЕНЦИАЛЬНО-АЛГЕБРАИЧЕСКИХ УРАВНЕНИЙ

Ю.Е. Гликликх

Мы исследуем дифференциально-алгебраические уравнения, возникающие в математических моделях некоторых радиотехнических устройств. Описан класс дифференциально-алгебраических уравнений, для которого доказаны необходимые и достаточные условия для глобального по времени существования решений. Как и во многих других работах, в которых получены достаточные условия для таких уравнений, мы сводим их к обыкновенным дифференциальным уравнениям и затем применяем необходимые и достаточные условия для последних. Мы рассматриваем системы, у которых матричный пучок регулярен и (для простоты) характеристический многочлен удовлетворяет условию ранг-степень. Мы также требуем выполнения некоторых дополнительных условий, который позволяют сводить дифференциально-алгебраические уравнения к дифференциальным.

Ключевые слова: дифференциально-алгебраические уравнения; глобальное существование решений; соответствующие отображения; полная риманова метрика.

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