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THE LYAPUNOV STABILITY OF THE CAUCHY–DIRICHLET PROBLEM FOR THE GENERALIZED HOFF EQUATION

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We consider the initial boundary value problem with homogeneous Dirichlet boundary conditions for the generalized Hoff equation in a bounded domain. This equation models the dynamics of buckling of a double-tee girder under constant load and belongs to a large class of Sobolev type semilinear equations (We can isolate the linear and non-linear parts of the operator acting on the original function). The paper addresses the stability of zero solution of this problem. There are two methods in the theory of stability: the first one is the study of stability by linear approximation and the second one is the study of stability by Lyapunov function. We use the second Lyapunov's method adapted to the case of incomplete normed spaces. The main result of this paper is a theorem on the stability and asymptotic stability of zero solution to this problem.

Keywords: Sobolev-type equation; phase space; Lyapunov stability.

Introduction

Let $\Omega \subset \mathbb{R}^s$ be a bounded domain with boundary $\partial\Omega$ of class C^∞. Consider the generalized Hoff equation [1] in cylinder $\Omega \times \mathbb{R}$

$$(\lambda - \lambda_0)u_t + \Delta u_t = \alpha_1 u + \alpha_2 u^3 + \ldots + \alpha_n u^{2n-1}, \quad n \in \mathbb{N}. \tag{1}$$

This equation models the bending of an I-beam. Here the function $u = u(x,t)$, $(x,t) \in \Omega \times \mathbb{R}$ is the displacement of the beam from the vertical position. The parameter $\lambda \in \mathbb{R}_+$ corresponds to a constant vertical load and the parameters $\alpha_i \in \mathbb{R}$, $i = 1,2,...,n$ characterize the material of the beam.

Consider the initial-boundary value problem

$$u(x,0) = u_0(x), x \in \Omega; \quad u(x,t) = 0, \quad (x,t) \in \partial\Omega \times \mathbb{R} \tag{2}$$

for equation (1). This problem was firstly considered in [2 – 4], wherein it was found out that the problem is essentialy unsolvable for arbitrary initial data. The set of initial values, which guarantees the existence and uniqueness of solution to initial-boundary value problem for equation (1), has been studied in [5]. If $n = 2$ and $\alpha_1 \cdot \alpha_2 \in \mathbb{R}_+$ then the phase space of equation (1) is a simple Banach C^∞-manifold. This result was obtained in [6]. And if $\alpha_1 \cdot \alpha_2 \in \mathbb{R}_-$ then the phase space of equation (1) lies on the Whitney fold. It is shown in [7]. The generalized Hoff equation (for $n > 3$) was considered in [8], but in this paper the stability has not been studied. This result was obtained in [9] for the case when $n = 3$. Generalized Sobolev type equations have been studied in other papers, for example, in [10]. In this paper we generalize the results of [9] and consider the case when $n > 3$.

The paper consists of two parts. The first part is devoted to the reduction of problem (1), (2) to the Cauchy problem

$$u(0) = u_0 \tag{3}$$

for abstract semilinear Sobolev type equation

$$L\dot{u} = Mu + N(u). \tag{4}$$

Here L, M are linear operators and N is nonlinear operator defined on specially constructed functional spaces. The second part is devoted to the study of stability of stationary solution to problem (1), (2). This is the main result.

1. Phase Space

Consider spaces $\mathfrak{U} = \overset{\circ}{W}{}_2^1(\Omega)$, $\mathfrak{F} = W_2^{-1}(\Omega)$ and operators

$$\langle Lu, v\rangle = \int\limits_\Omega ((\lambda - \lambda_0)uv - \nabla u \nabla v)dx, \quad \forall u, v \in \overset{\circ}{W}{}_2^1(\Omega), \tag{5}$$

$$\langle Mu, v\rangle = \alpha_1 \int\limits_\Omega uvdx, \quad \forall u, v \in L_{2n}(\Omega), \tag{6}$$

$$\langle N(u), v\rangle = \int\limits_\Omega (\alpha_2 u^3 + ... + \alpha_{n-1}u^{2n-3} + \alpha_n u^{2n-1})vdx \quad \forall u, v \in L_{2n}(\Omega). \tag{7}$$

Embedding $\overset{\circ}{W}{}_2^1(\Omega) \hookrightarrow L_{2n}(\Omega)$ is dense and continuous. Therefore $L, M \in \mathcal{L}(\mathfrak{U}; \mathfrak{F})$, and L is a Fredholm operator. The spectrum of the operator L is real and discrete, has finite multiplicity and condenses only to $-\infty$. Operator $N \in C^\infty(\mathfrak{U}; \mathfrak{F})$.

A vector function $u \in C^\infty(\mathfrak{U}, \mathfrak{F})$ satisfying equation (4) is called a solution of this equation.

Definition 1. A set $\mathfrak{P} \subset \mathfrak{U}$ is called a *phase space* of equation (3) if the following conditions are satisfied:

(i) each solution $u = u(t)$ of (4) lies in $\mathfrak{P}$ pointwisely; i.e., $u(t) \in \mathfrak{P}$ for all $t \in \mathbb{R}$;

(ii) for each $u_0 \in \mathfrak{P}$, there exists a unique solution to problem (3), (4).

Theorem 1. *Let $n = 1, 2$ when $s = 4$ (i.e. $\Omega \subset \mathbb{R}^4$), $n = 1, 2, 3$ when $s = 3$ (i.e. $\Omega \subset \mathbb{R}^3$) and $n \in \mathbb{N}$ when $s = 1, 2$ (i.e. $\Omega \subset \mathbb{R}$ or $\Omega \subset \mathbb{R}^2$). Then one of the two conditions is satisfied*

(i) *if* $\ker L = \{0\}$, *then the phase space of equation (1) concides with* $\mathfrak{U}$.

(ii) *if* $\ker L \neq \{0\}$, *and all coefficients $\alpha_i \in \mathbb{R}\backslash\{0\}$, $i = 1, ..., n$ have the same sign. Then the phase space of eqiation (1) is simple manifold*

$$\mathfrak{M} = \left\{ u \in \mathfrak{U} : \int\limits_\Omega (\alpha_1 + \alpha_2 u^2 + ... + \alpha_n u^{2n-2}) u \chi_k dx = 0,\ k = 1, \ldots, m \right\}.$$

Here χ_k are orthonormal eigenfunctions corresponding to the eigenvalues λ_k of L.

2. Stability

Definition 2. A family of mappings S is called a *nonlinear semigroup* in a normed space $\mathfrak{V}$ if for every $u \in \mathfrak{V}$ and some $\tau = \tau(u) \in \mathbb{R}_+$ the following conditions are satisfied:

(i) $S = S(t,u) \in \mathfrak{V}$, for all $t \in (-\tau;\tau)\,;S\,(0,u) = u$;

(ii) $S(t+s,u) = S(t, S(s,u))$ for all $t+s \in (-\tau,\tau)$.

A point $u \in \mathfrak{V}$ such that $S(t,u) = u, t \in \mathbb{R}$ is called a *stationary point.*

Definition 3. A stationary point u is called

(i) *stable* (in sense of A.M. Lyapunov), if for any neighborhood $\mathfrak{O}_u$ of u there exists a neighborhood $\mathfrak{O}'_u$ (i.e. not necessarily the same neighborhood) of the same point, such that $S(t,v) \in \mathfrak{O}'_u$ for all $v \in \mathfrak{O}_u$ and $t \in \mathbb{R}_+$;

(ii) *asymptotically stable* (in sense of A.M. Lyapunov), if it is stable and for any point v in some neighborhood $\mathfrak{O}_u$ of u $S(t,v) \to u$ for $t \to \infty$.

Definition 4. A functional $V \in C(\mathfrak{V};\mathbb{R})$ is called a *Lyapunov functional* if

$$\dot{V}(u) = \overline{\lim_{t\to 0+}}\frac{1}{t}\left(V(S(t,u)) - V(u)\right) \le 0$$

for all $u \in \mathfrak{V}$.

Theorem 2. *Let u be a stationary point. If there exists a Lyapunov functional such that*

(i) $V(u) = 0$*;*

(ii) $V(v) \ge \varphi(\|v-u\|)$*; here φ is strictly increasing continuous function such that $\varphi(0) = 0$ and $\varphi(r) > 0$ for $r \in \mathbb{R}_+$, then the point u is stable.*

Theorem 3. *Let the conditions of Theorem 2 be satisfied, and a strictly increasing continuous function ψ, such that $\psi(0) = 0$ and $\psi(r) > 0$ for $r \in \mathbb{R}_+$, exist. If $\dot{V}(v) \le -\psi(\|v-u\|)$, then the point u is asymptotically stable.*

We will study the stability of problem (1),(2) using Theorems 2 and 3.

Consider a space $\mathfrak{U}\,(=\overset{\circ}{W}{}^1_2)$ with norm $\|\cdot\|$ of space L_2. It is an incomplete normed space. Define the Lyapunov functional by formula

$$V(u) = \int\limits_\Omega (u_x^2 + (\lambda_0 - \lambda)u^2)dx.$$

Obviously $V(0) = 0$ and $V(u) \ge c\|u\|^2$. Moreover multiplying (1) scalarly in L_2 by u we obtain

$$\dot{V}(u) = -\alpha_1\|u\|^2 - \alpha_2\|u\|^4_{L_4} - \ldots - \alpha_n\|u\|^{2n}_{L_{2n}}.$$

Since the embedding $L_{2n} \hookrightarrow L_2$ is obvious, the following inequality holds

$$\dot{V}(u) \le -\alpha_1\|u\|^2 - \alpha_2 c_2\|u\|^4_{L_4} - \ldots - \alpha_n c_n\|u\|^{2n}_{L_{2n}}. \tag{8}$$

Here $c_j, j = \overline{1,n}$ are the constants of embedding. Function constructed from the norm $\|\cdot\|$ on the right side of (8) satisfies the conditions of Theorem 3. So we have proved the following theorem.

Theorem 4. *Zero solution of problem (1), (2) is asymptotically stable for any* $\alpha_j \in \mathbb{R}_+$, $j = \overline{1,n}$, $\lambda \in [0, \lambda_0]$.

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УСТОЙЧИВОСТЬ ПО ЛЯПУНОВУ ЗАДАЧИ КОШИ – ДИРИХЛЕ ДЛЯ ОБОБЩЕННОГО УРАВНЕНИЯ ХОФФА

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В данной статье исследуется начально-краевая задача Коши с однородными граничными условиями Дирихле для обобщенного уравнения Хоффа, заданного в ограниченной области. Это уравнение моделирует динамику выпучивания двутавровой балки, находящейся под постоянной нагрузкой и относится к классу полулинейных (у оператора действующего на исходную функцию можно выделить линейную часть и нелинейную) уравнений соболевского типа. Нас интересует устойчивость нулевого решения данной задачи. В рамках теории устойчивости выделяют два метода: первый — исследование устойчивости по линейному приближению и второй — исследование устойчивости посредством функции Ляпунова. Отметим, что первым методом Ляпунова исследовать устойчивость решения уравнения Хоффа, заданного в области, не удается, поскольку в нашем случае относительный спектр оператора M пересекается с мнимой осью. Поэтому для нашей задачи был применен метод функций Ляпунова, модифицированный для случая неполных нормированных пространств. В результате получена теорема об устойчивости и асимптотической устойчивости нулевого решения данной задачи.

Ключевые слова: уравнение соболевского типа; фазовое пространство; устойчивость по Ляпунову.

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